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The Small Free Vibrations and Deformation of a Thin Elastic Shell

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XVI. *The Small Free Vibrations and Deformation of a Thin Elastic Shell.*

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Contents.

§ 1.—*Historical Introduction.*

I propose, in the first place, to give a brief account of the principal theories of the vibrations and flexure of a thin elastic plate hitherto put forward, and afterwards to apply the method of one of them to the case when the plate in its natural state has finite curvature.

Passing over the early attempts of Mdlle. Sophie Germain, the first mathematician who succeeded in obtaining a theory of the flexure of a thin plane plate was Poisson. In his memoir * he obtains the differential equation for the deflection of the plate, which is generally admitted, and certain boundary-conditions, which have met with less general acceptance. The idea of Poisson's method may be simply stated. The plate being very thin, we may expand all the functions which occur in the equations of equilibrium and boundary-conditions in powers of the variable expressing the distance of a particle from the middle-surface in the natural state, then, taking only the terms up to the third order, we obtain the differential equations for the determination of the displacements which are generally admitted. The meaning of Poisson's boundary-conditions is as follows †:—Suppose the plate to form part of an infinite

* "Mémoire sur l'Équilibre et le Mouvement des Corps élastiques," 'Paris Acad. Mém.,' 1829.

† *Cf.* Thomson and Tait, 'Natural Philosophy,' part 2, pp. 188–9.

plate, and to be held in its actual position, partly by the forces directly applied to its mass, and partly by the action of the remainder of the plate exerted across the boundary; if the plate be now cut out, it will be necessary, in order to hold it in the same configuration, to apply at every point of its edge a distribution of force and couple identical with that exerted by the remainder before the plate was cut out. Now, it has been shown by KIRCHHOFF* that these equations express too much, and that it is not generally possible to satisfy them; but the method proposed by THOMSON and TAIT† gives a rational explanation of KIRCHHOFF's union of two of POISSON's boundary-conditions in one, and renders his theory complete. However, the objection raised by DE ST. VENANT‡ to the fundamental assumption that the stresses and strains in an element can be expanded in integral powers of the distance from the middle-surface, seems to require a different theory.

The next epoch in the theory of plates is marked by KIRCHHOFF's memoir just referred to. The method rests on two assumptions, viz.: (1) Every straight line of the plate which was originally perpendicular to the plane bounding surfaces remains straight after the deformation, and perpendicular to the surfaces which were originally parallel to the plane bounding surfaces; (2) all the elements of the middle-surface (*i.e.*, the surface which in the natural state was midway between the plane parallel bounding surfaces) remain unstretched. Both these assumptions may be shown to be approximately true in the cases of flexure and transverse vibration, but, as assumptions, they appear unwarrantable. In this memoir of KIRCHHOFF's the union of two of POISSON's boundary-conditions in one was first effected, the method employed to obtain the equations being that of virtual work. The theory of this memoir will be referred to as KIRCHHOFF's "first theory."

KIRCHHOFF§ has given a general method for the treatment of elastic bodies, some of whose dimensions are indefinitely small in comparison with others. In this method we consider, in the first place, the equilibrium of an element of the body all whose dimensions are of the same order as the indefinitely small dimensions. When we know the potential energy due to the internal strain of such an element, we obtain by integration over the remaining dimensions the whole potential energy due to the elastic strain of the body. Then, taking into account all the forces which act on the body, we can form the equation of virtual work, which will lead directly to the differential equations and boundary-conditions of our problem.

In KIRCHHOFF's method it appears that, to a first approximation, the bodily forces produce displacements which are negligible compared with those produced by the surface-tractions exerted upon the element by contiguous elements, and that, to the

* "Ueber das Gleichgewicht und die Bewegung einer elastischen Scheibe," 'CRELLE, Journ. Math.,' vol. 40.

† *Loc. cit.*, pp. 190–1.

‡ Translation of CLEBSCH's 'Elasticität,' Note on § 73, p. 725.

§ 'Vorlesungen über Mathematische Physik,' pp. 406 *et seq.*

same order of approximation, the displacements, when divided by finite quantities of one dimension in length, are negligible compared with the strains.

The application of this method to the theory of plates appears to have been first made by GEHRING, a pupil of KIRCHHOFF'S, at the latter's request; and the results will be found in KIRCHHOFF'S thirtieth lecture, and in CLEBSCH'S 'Theorie der Elasticität fester Körper,' §§ 64 *et seq.* We shall call the theory thus deduced KIRCHHOFF'S "second theory." POISSON and KIRCHHOFF had both arrived at the equations S, T, R = 0,* which express that the traction exerted on an element of a surface normal to the middle-surface of the plate is everywhere tangential to the middle-surface. These equations are fundamental in KIRCHHOFF'S second theory. This appears to lie at the root of the objection raised by DE ST. VENANT† to this theory, as it is stated by him that S and T, if they exist, may produce important effects, especially when the material of the plate is not isotropic.

It seems unnecessary to explain in detail THOMSON and TAIT'S treatment of the problem. We need only note here that the equations S, T, R = 0 are a basis for this theory also.

[*Added July*, 1888.—An important inference from the method is that a line of particles initially normal to the middle-surface is approximately normal to this surface after strain. This is expressed by the vanishing of the shears a and b, as given by equations (11) *infra*. This conclusion is intimately bound up with the conclusion that S and T vanish. At the edge of the plate S and T may have given values which do not vanish, and the approximate perpendicularity of line-elements originally perpendicular to the middle-surface will here break down. The transition from a state of things in which S and T exist at the edge to one in which they vanish, on a surface parallel to the edge and very near to it, is illustrated by the discussion in THOMSON and TAIT'S 'Natural Philosophy,' §§ 721–729. The conclusion seems to be that KIRCHHOFF'S general method for the treatment of elastic bodies, some of whose dimensions are indefinitely small in comparison with others, cannot be applied to the elements situated very near to the edge of a plate, as the strain is not produced in these by the action of contiguous elements. We may, nevertheless, regard it as giving correctly, not only the potential energy due to the strain of an element at a distance from the edge, but also the whole potential energy arising from the strain in all the elements. It will thus lead us to the right differential equations of motion or equilibrium and boundary-conditions.]

The theory of the flexure of an elastic plate has been placed in a much clearer light by the researches of BOUSSINESQ, who has treated the subject in a masterly manner in two memoirs. In the first of these‡ he has certainly proved that S = 0, T = 0, R = 0 is an approximation to the actual state of stress within an element of the

* I use THOMSON and TAIT'S notation for the stresses, strains, and elastic constants.

† Translation of CLEBSCH'S 'Elasticität,' p. 691.

‡ 'LIOUVILLE, Journal de Math.,' 1871.

plate; and he says that R = 0 to a higher degree of approximation than S and T. Taking $2h$ for the thickness of the plate, and the plane of xy for the middle-surface in the natural state, we have, on integration, with reference to z,

$$T = -\int_{-h}^{h} \left(\rho X + \frac{\partial P}{\partial x} + \frac{\partial U}{\partial y}\right) dz,$$

$$S = -\int_{-h}^{h} \left(\rho Y + \frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y}\right) dz,$$

$$R = -\int_{-h}^{h} \left(\rho Z + \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y}\right) dz.$$

Assuming that the bodily forces are not such that if applied to a body of finite size they would produce deformations indefinitely great compared with those produced in the plate, and that P, Q, U do not vary rapidly from one element to another, we see that S, T, R are small compared with P, Q, U. BOUSSINESQ proceeds to express three of the strains in terms of the rest by means of the relations S, T, R = 0, as was done in KIRCHHOFF's second theory; then, by means of these approximate values, he finds S, T to a higher order, and on substituting in the general equations of equilibrium obtains the well-known equation for the deflection of the plate. The method of securing the union of two of POISSON's boundary-conditions in one is the same as that previously given by THOMSON and TAIT.

BOUSSINESQ returned to the subject in 1879, in a second memoir published in 'LIOUVILLE's Journal.' Apparently dissatisfied with the assumptions S, T = 0, he proposed to consider the subject in the following manner. Let the plate be divided into similar elementary rectangular prisms, whereof the linear dimensions are all comparable, and suppose these prisms bounded by the plane surfaces of the plate, and by pairs of parallel planes at right angles to these surfaces. Two neighbouring prisms must always be in nearly the same condition as regards strain, except in the case of prisms situated near the edge. Hence, generally, the component stresses will be approximately the same at all points on the same surface parallel to the middle-surface, and not infinitely near the edge of the plate. Hence, in this kind of equilibrium, the stresses will be approximately independent of the position on the middle-surface of the centre of the element. This is precisely KIRCHHOFF's result* deduced from the kinematics of the system, and it appears certainly true when the plate is very thin. BOUSSINESQ wishes his theory to apply to plates of small finite thickness, and he proposes to replace the equations just found by the following

$$R = 0, \qquad \frac{\partial T}{\partial x} = 0, \qquad \frac{\partial T}{\partial y} = 0, \qquad \frac{\partial S}{\partial x} = 0, \qquad \frac{\partial S}{\partial y} = 0, \qquad \frac{\partial^2}{\partial x^2, \partial y^2, \partial x\, \partial y}(P, Q, U) = 0;$$

* 'Vorlesungen,' p. 453, remark on equations (8).

these suppositions are more general than those of the former paper, and enable the author to take account better of the effects of æolotropy of the material of the plate.

DE ST. VENANT* obtains the equations for flexure on the assumptions (1) that R = 0, (2) that the middle-surface of the plate is bent without stretching, so that the extension of any line-element through a point distant z from the middle surface and parallel thereto is z/ρ, where ρ is the radius of curvature of the normal section of the middle-surface through a line parallel to the element. From these suppositions, of which the first is justified in the manner of BOUSSINESQ's memoirs, the ordinary equations are deduced and extended to the case of æolotropic plates. From the inapplicability of the second of these suppositions to the case when the plate is initially curved† we may be justified in denying it the right to be a foundation for the theory.

The question between the methods of KIRCHHOFF's second theory and BOUSSINESQ's memoirs may be taken to be that of the degree of approximation obtainable by the former. It seems to be established that the terms which occur in CLEBSCH's equations‡ are correct to the order of approximation adopted; but the question arises whether, if it were desirable to obtain a higher degree of approximation in the equations, this could be effected by means of KIRCHHOFF's second theory; and it appears that, so long as the equations S, T = 0 are retained with R = 0 for the purpose of giving three of the strains in terms of the rest, this question must be answered in the negative. It must be observed that KIRCHHOFF only uses these equations for this purpose, just as BOUSSINESQ does in his first memoir, while the equations and conditions are found by applying the principle of virtual work.

In a recent paper§ I have proposed a modification of KIRCHHOFF's second theory, with the view of showing how his kinematical equations, whose accuracy has been disputed by BOUSSINESQ, can be made exact. The equations referred to are those unnumbered on page 452 of the 'Vorlesungen.' In these certain differential coefficients are introduced, and afterwards neglected as small; and BOUSSINESQ has contended that they should be retained. In the paper referred to I have endeavoured to show that these differential coefficients have no meaning so long as we are treating the equilibrium of an elementary portion of the plate, all whose dimensions are of the same order as the thickness, so that the equations can be made exact by simply omitting these differential coefficients. As will hereafter appear, KIRCHHOFF's process applies directly to the theory of a thin elastic shell, and the modification proposed in the theory of plates has place equally in that of shells. This will be fully explained in the sequel (Art. 2).

* Translation of CLEBSCH. Note to § 73.

† This will be proved in the sequel.

‡ 'Elasticität,' pp. 306, 307, equations (105) and (106).

§ "Note on KIRCHHOFF's Theory of the Deformation of Elastic Plates," 'Cambridge Phil. Soc. Proc.,' vol. 6, 1887.

§ 2. *Theory of Shells.*

In this paper the potential energy of deformation of an isotropic elastic shell is investigated by the same method as that employed by KIRCHHOFF in his discussion of the vibrations of a plane plate.* The shell is supposed to be bounded by two surfaces parallel to its middle-surface, and is deformed in any arbitrary manner. The expression given by KIRCHHOFF for the energy of the plate per unit area of its middle-surface is

$$\tfrac{2}{3}\mathrm{K}h^3\left[q_1^2 + p_2^2 + 2p_1^2 + \frac{\theta}{1+\theta}(q_1 - p_2)^2\right]$$
$$+ 2\mathrm{K}h\left[\sigma_1^2 + \sigma_2^2 + \tfrac{1}{2}\tau^2 + \frac{\theta}{1+\theta}(\sigma_1 + \sigma_2)^2\right]^{\dagger}$$

where $2h$ is the thickness of the plate, K the rigidity, and $\theta/(1+\theta) = \sigma/(1-\sigma)$, σ being the ratio of linear lateral contraction to linear longitudinal extension of the material; σ_1, σ_2 are the extensions of two line-elements of the middle-surface initially at right angles, and τ the complement of the angle between them after strain; q_1, p_2, p_1 are quantities defining the curvature of the middle-surface after strain, viz.:—

$$p_2 - q_1 = \text{sum of principal curvatures,}$$
$$-(p_2 q_1 + p_1^2) = \text{measure of curvature};$$

so that, if ρ_1, ρ_2 be the principal radii of curvature after strain, the first term of the above reduces to

$$\tfrac{2}{3}\mathrm{K}h^3\frac{1+2\theta}{1+\theta}\left[\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)^2 - \frac{1+\theta}{1+2\theta}\,\frac{1}{\rho_1\rho_2}\right].$$

A similar expression to that given by KIRCHHOFF is obtained below in the case of the shell initially curved; but here the quantities q_1, p_2, p_1 are replaced by the difference of their values in the strained and unstrained states, a result which might have been anticipated from the remarks made by KIRCHHOFF ('Vorlesungen,' p. 413) on the strain of a rod initially curved, since the strain of an element is a linear function of these quantities.

We wish to obtain equations of motion and boundary-conditions in terms of the displacements of a point on the middle-surface of the shell, these being reckoned parallel to the lines of curvature and perpendicular to the tangent plane at the point. For this purpose it is necessary to express all the quantities which occur in the potential-energy-function in terms of these displacements. As the geometrical theory of the deformation of extensible surfaces appears not to have been hitherto made out,

* Called above "KIRCHHOFF'S second theory."

† 'Vorlesungen,' p. 454.

it was necessary to give the elements of such a theory for small deformations. The expressions obtained for the principal radii of curvature show that the potential energy due to bending is never the same quadratic function of the changes of principal curvature as for a plane plate, except in the single case where the middle-surface is a sphere and unstretched.

The general variational equation of motion is developed in the form of surface and line integrals, and the equations reduce to those of CLEBSCH * in the case of a plane plate. The terms herein which depend on externally applied forces are obtained directly, without the use of the arbitrary multipliers which render the calculations of CLEBSCH so tedious, and without the necessity which he finds for correcting an "error" † as regards the distribution of force at the edge, thus avoiding some of the criticisms of DE ST. VENANT.‡

We know that when a plane plate vibrates the transverse displacement is independent of the displacements parallel to the plane of the plate; and when the transverse vibrations alone are taking place no line on the middle-surface is altered in length. I discuss the question whether vibrations of the shell are possible in which this last condition holds good, and show that it leads to three partial differential equations giving the displacements as functions of the position of a point on the middle-surface, and that these equations are not in general of a sufficiently high order to admit of solutions which shall also satisfy the conditions which hold at a free edge. This result is quite independent of the theory adopted, as the equations of inextensibility are in the most general case a system of the third order, while the boundary-conditions are four in number. It would, of course, be possible to find a system of forces applied to the boundary which could artificially maintain this kind of vibration. It appears, then, that the term of the potential energy which depends on the bending, which is multiplied by h^3, is small compared with the term depending on the stretching, which is multiplied by h; and, in order to obtain the limiting form of the theory when $h=0$, we may form approximate equations of equilibrium and motion and boundary-conditions by omitting the term in h^3. Having formed these equations, I proceed to discuss the question whether the shell can execute vibrations in which there shall be no tangential displacement, and it is shown that this requires both the principal radii of curvature of the middle-surface to be constant at every point. The frequencies of the purely radial vibrations of a sphere and an infinitely long circular cylinder are given; the displacement is a simple harmonic function of the time, and is the same at all points of the sphere or cylinder. The formula for the frequency admits of independent verification. Another general result deduced from the approximate equations is that any shell whose middle-surface is a surface of revolution

* 'Elasticität,' pp. 306, 307; Equations (105), (106).

† *Ibid.*, p. 284.

‡ Translation of CLEBSCH, p. 691. The method of CLEBSCH is styled "obscure, indirecte, fort compliquée."

can execute purely tangential vibrations such that every point moves perpendicularly to the meridian through it, and the displacement is symmetrical about the axis of revolution.

The special problem of the vibrations of a spherical shell has been discussed by Lord RAYLEIGH.* In his paper it is assumed that no line on the middle-surface is altered in length; the boundary-conditions are not considered. The form of the potential energy taken is a quadratic function of the changes of principal curvature or the middle-surface, and this I have proved to be in this case the true form in Art. 7. The assumption of inextensibility does in this case lead to expressions for the displacements which cannot satisfy the boundary-conditions which hold at a free edge.

The method developed in this paper is applied to the problem, and the approximate equations integrated. The solution comes out in tesseral harmonics with fractional or imaginary indices, and the frequency is given by a transcendental equation; in case the shell be hemispherical this equation is simplified, and to express the symmetrical vibrations only the ordinary zonal harmonics with real integral indices are required, and the frequency equation can be solved.

As a further example of the application of the method to small vibrations I have discussed the vibrations of a cylindrical shell. The displacement of a point on the middle-surface is expressed by simple harmonic functions of the cylindrical coordinates of the point. In the case of the symmetrical vibrations the frequency equation is easily solved.

ARON has applied the method of CLEBSCH to the problem of shells. In his memoir† a point on the middle-surface of the shell is considered as defined by two parameters, as in GAUSS's theory of the curvature of surfaces; the displacements are referred to an arbitrary system of fixed axes; and the expressions found for them are the same as those in Art. 4 of this paper, but the work contains a small error (see note to Art. 4). Free use is made of arbitrary multipliers in order to obtain the equations of equilibrium referred to the fixed axes. As these are in a very unmanageable shape, a method of forming equations referred to moving axes is indicated; the equations are first obtained with reference to fixed axes, and it is proposed to transform these. The transformation is not effected, but some reductions are made with a view to it (pp. 169 *et seq.*). In these reductions all effects due to the change of direction of the axes as we go from point to point on the middle-surface are neglected, so that the results are erroneous (see note to Art. 6).

A theory of the vibrations of a shell whose middle-surface is a surface of revolution has been given by MATHIEU.‡ The method is similar to that employed by POISSON for the plate, viz., taking $\gamma = 0$ for the middle-surface, all the quantities which occur

* "On the Infinitesimal Bending of Surfaces of Revolution," 'London Math. Soc. Proc.,' vol. 17, 1882.

† "Das Gleichgewicht und die Bewegung einer unendlich dünnen beliebig gekrümmten elastischen Schale." 'CRELLE, Journ. Math.,' vol. 78, 1874, p. 138.

‡ "Mémoire sur le Mouvement vibratoire des Cloches," 'Journ. de l'École Polytechn.,' cahier 51 (1883).

are expanded in powers of γ, and approximate equations taken. These equations are included in those given in the present paper for shells whose middle-surface is any whatever. MATHIEU gives for the special case some of the theorems on purely normal and purely tangential vibrations here proved (see notes to Art. 13). The solution for spherical shells is given in his paper. The introduction of the generalised tesseral harmonic into this solution enables us to recognise that a certain type of vibration given by MATHIEU cannot exist (see note to Art. 18). The objections raised by DE ST. VENANT to POISSON's method for plates seem to lie equally against its extension to shells.

§ 3. *Internal Strain in an Element of the Shell.*

1. Suppose the lines of curvature on the middle-surface of the shell to be drawn; let these be the curves $\alpha=$ const., $\beta=$ const.; then any point on the middle-surface is given by its α, β. At each intersection of a curve α with a curve β suppose the normal to the middle-surface drawn and lengths h marked off upon it inwards and outwards from the surface, the *loci* of the extremities of these lines are two surfaces parallel to the middle-surface. If we suppose the space between these surfaces filled with isotropic elastic material we obtain the elastic solid shell which we wish to treat.

Let the middle-surface be covered with a network of the lines $\alpha=$ const., $\beta=$ const. at distances from each other comparable with the thickness of the shell, and suppose the normals drawn as above described at all the points of these curves. The shell will thus be divided into a great number of elementary prisms; and, according to KIRCHHOFF's general method, we must first discuss the equilibrium of one of these elementary prisms.

Let α, β be the parameters of the centre P of one of these elementary prisms before strain. Imagine three line-elements of the shell (1, 2, 3) to proceed from P, the elements (1) and (2) being along the lines β, α through P, and (3) along the normal at P to the middle-surface. Then after strain these lines are not in general co-orthogonal, but by means of them we can construct a system of rectangular axes to which we can refer points in the prism whose centre is P. Thus, P is to be the origin, the axis of x is to lie along the line-element (1), and the plane of x, y is to contain the line-elements (1) and (2); then the line-element (2) will make an indefinitely small angle with the axis y, and the line-element (3) will make an indefinitely small angle with the axis z.

By means of the lines of curvature and the middle-surface we can construct a system of orthogonal surfaces (α, β, γ), so that we may use the formulæ of orthogonal coordinates with reference to α, β, γ.

We write for the distance between two near points—

$$\left[\left(\frac{d\alpha}{h_1}\right)^2+\left(\frac{d\beta}{h_2}\right)^2+\left(\frac{d\gamma}{h_3}\right)^2\right]^{\frac{1}{2}}.$$

2. The point P is defined before the strain by its α, β, and lies on a certain surface $\gamma = 0$ (the middle-surface). The prism whose centre is P is held in equilibrium by the action of adjacent prisms, and its parts are not in the same configuration as that in which they would be found if this prism were separated from the rest of the shell and left to itself.* Now, if this portion were isolated from the action of neighbouring portions, any point of it (Q) would take a certain position defined by the intersection of three surfaces of the family (α, β, γ), which we may take to be $\alpha + p, \beta + q, r$. Hence, when this prism is subject to the action of neighbouring prisms the position of Q will be given with reference to the (x, y, z) axes at P by $p/h_1 + u_0$, $q/h_2 + v_0$, $r/h_3 + w_0$, and after the strain is effected it will be given by $p/h_1 + u'$, $q/h_2 + v'$, $r/h_3 + w'$ referred to the axes of (x, y, z) defined in Art. 1. The component displacements (u_1, v_1, w_1) of Q are $u' - u_0$, $v' - v_0$, $w' - w_0$.

Consider a system of rectangular axes fixed in space, and after strain let ξ, η ζ, be the coordinates of P referred to this system, and let the directions of the (x, y, z) axes be connected with those of the fixed (ξ, η, ζ) axes by the scheme—

	ξ	η	ζ
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

Then, after strain the coordinates of Q are

$$\left.\begin{aligned} &\xi + l_1\left(\frac{p}{h_1} + u'\right) + l_2\left(\frac{q}{h_2} + v'\right) + l_3\left(\frac{r}{h_3} + w'\right), \\ &\eta + m_1\left(\frac{p}{h_1} + u'\right) + m_2\left(\frac{q}{h_2} + v'\right) + m_3\left(\frac{r}{h_3} + w'\right), \\ &\zeta + n_1\left(\frac{p}{h_1} + u'\right) + n_2\left(\frac{q}{h_2} + v'\right) + n_3\left(\frac{r}{h_3} + w'\right). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

These expressions are functions of $\alpha + p$, $\beta + q$, r; and, hence, for each of them we have $\partial/\partial\alpha = \partial/\partial p$ and $\partial/\partial\beta = \partial/\partial q$. In forming these differential coefficients it is important to observe that u', v', w' have no differential coefficients with respect to α, β. Throughout the space within which u', v', w' exist, viz., the range of values of p, q, r, which correspond to points within the elementary prism treated, α, β do not vary. In his theory KIRCHHOFF first introduces the differential coefficients analogous to

* This remark was made by ARON, in his memoir in BORCHARDT'S (CRELLE'S) 'Journal,' vol. 78, p. 138.

$\partial u'/\partial\alpha$ . . ., and afterwards neglects them as small. So that the equations (6) and (7) to be obtained below are unaffected by the modification of the theory here proposed.

Equating the differential coefficients of (1) with respect to α and p, we get

$$\frac{\partial\xi}{\partial\alpha}+\frac{\partial l_1}{\partial\alpha}\left(\frac{p}{h_1}+u'\right)+\frac{\partial l_2}{\partial\alpha}\left(\frac{q}{h_2}+v'\right)+\frac{\partial l_3}{\partial\alpha}\left(\frac{r}{h_3}+w'\right)+l_1 p\frac{\partial}{\partial\alpha}\left(\frac{1}{h_1}\right)+l_2 q\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)$$
$$+\,l_3 r\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)=l_1\left(\frac{1}{h_1}+\frac{\partial u'}{\partial p}\right)+l_2\frac{\partial v'}{\partial p}+l_3\frac{\partial w'}{\partial p},$$

$$\frac{\partial\eta}{\partial\alpha}+\frac{\partial m_1}{\partial\alpha}\left(\frac{p}{h_1}+u'\right)+\frac{\partial m_2}{\partial\alpha}\left(\frac{q}{h_2}+v'\right)+\frac{\partial m_3}{\partial\alpha}\left(\frac{r}{h_3}+w'\right)+m_1 p\frac{\partial}{\partial\alpha}\left(\frac{1}{h_1}\right)+m_2 q\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)$$
$$+\,m_3 r\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)=m_1\left(\frac{1}{h_1}+\frac{\partial u'}{\partial p}\right)+m_2\frac{\partial v'}{\partial p}+m_3\frac{\partial w'}{\partial p},$$

$$\frac{\partial\zeta}{\partial\alpha}+\frac{\partial n_1}{\partial\alpha}\left(\frac{p}{h_1}+u'\right)+\frac{\partial n_2}{\partial\alpha}\left(\frac{q}{h_2}+v'\right)+\frac{\partial n_3}{\partial\alpha}\left(\frac{r}{h_3}+w'\right)+n_1 p\frac{\partial}{\partial\alpha}\left(\frac{1}{h_1}\right)+n_2 q\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)$$
$$+\,n_3 r\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)=n_1\left(\frac{1}{h_1}+\frac{\partial u'}{\partial p}\right)+n_2\frac{\partial v'}{\partial p}+n_3\frac{\partial w'}{\partial p};$$

and, similarly, by differentiating with respect to β and q.

3. Now, taking the set of three equations above written, multiply them by l_1, m_1, n_1 and add, then by l_2, m_2, n_2 and add, then by l_3, m_3, n_3 and add, and repeat the process on the second set; the six resulting equations may be written

$$\left.\begin{aligned}\frac{1}{h_1}+\frac{\partial u'}{\partial p}&=\frac{1+\sigma_1}{h_1}+\frac{1}{h_1}\lambda'_1\left(\frac{r}{h_3}+w'\right)-\frac{1}{h_1}\tau'_1\left(\frac{q}{h_2}+v'\right)+p\frac{\partial}{\partial\alpha}\left(\frac{1}{h_1}\right)\\ \frac{\partial v'}{\partial p}&=\frac{1}{h_1}\tau'_1\left(\frac{p}{h_1}+u'\right)-\frac{1}{h_1}\kappa'_1\left(\frac{r}{h_3}+w'\right)+q\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\\ \frac{\partial w'}{\partial p}&=\frac{1}{h_1}\kappa'_1\left(\frac{q}{h_2}+v'\right)-\frac{1}{h_1}\lambda'_1\left(\frac{p}{h_1}+u'\right)+r\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)\end{aligned}\right\},\quad .\quad (2)$$

and

$$\left.\begin{aligned}\frac{\partial u'}{\partial q}&=\frac{\varpi}{h_2}+\frac{1}{h_2}\lambda'_2\left(\frac{r}{h_3}+w'\right)-\frac{1}{h_2}\tau'_2\left(\frac{q}{h_2}+v'\right)+p\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\\ \frac{1}{h_2}+\frac{\partial v'}{\partial q}&=\frac{1+\sigma_2}{h_2}+\frac{1}{h_2}\tau'_2\left(\frac{p}{h_1}+u'\right)-\frac{1}{h_2}\kappa'_2\left(\frac{r}{h_3}+w'\right)+q\frac{\partial}{\partial\beta}\left(\frac{1}{h_2}\right)\\ \frac{\partial w'}{\partial q}&=\frac{1}{h_2}\kappa'_2\left(\frac{q}{h_2}+v'\right)-\frac{1}{h_2}\lambda'_2\left(\frac{p}{h_1}+u'\right)+r\frac{\partial}{\beta\partial}\left(\frac{1}{h_3}\right)\end{aligned}\right\}.\quad .\quad (3)$$

In these σ_1, σ_2 are the extensions of the line-elements (1), (2), and ϖ is the sine of the angle the axis y makes with the line-element (2) after strain, so that, if (L_2, M_2, N_2) be the direction cosines of the line-element (2) after strain referred to the fixed axes of (ξ, η, ζ),

$$\left.\begin{aligned} l_1(1+\sigma_1) &= h_1\frac{\partial\xi}{\partial\alpha}, & m_1(1+\sigma_1) &= h_1\frac{\partial\eta}{\partial\alpha}, & n_1(1+\sigma_1) &= h_1\frac{\partial\zeta}{\partial\alpha} \\ \mathrm{L}_2(1+\sigma_2) &= h_2\frac{\partial\xi}{\partial\beta}, & \mathrm{M}_2(1+\sigma_2) &= h_2\frac{\partial\eta}{\partial\beta}, & \mathrm{N}_2(1+\sigma_2) &= h_2\frac{\partial\zeta}{\partial\beta} \\ \mathrm{L}_2 &= l_2 + l_1\varpi, & \mathrm{M}_2 &= m_2 + m_1\varpi, & \mathrm{N}_2 &= n_2 + n_1\varpi \end{aligned}\right\}. \quad (4)$$

Also

$$\left.\begin{aligned} \kappa'_1 &= h_1\left(l_3\frac{\partial l_2}{\partial\alpha} + m_3\frac{\partial m_2}{\partial\alpha} + n_3\frac{\partial n_2}{\partial\alpha}\right), & \kappa'_2 &= h_2\left(l_3\frac{\partial l_2}{\partial\beta} + m_3\frac{\partial m_2}{\partial\beta} + n_3\frac{\partial n_2}{\partial\beta}\right) \\ \lambda'_1 &= h_1\left(l_1\frac{\partial l_3}{\partial\alpha} + m_1\frac{\partial m_3}{\partial\alpha} + n_1\frac{\partial n_3}{\partial\alpha}\right), & \lambda'_2 &= h_2\left(l_1\frac{\partial l_3}{\partial\beta} + m_1\frac{\partial m_3}{\partial\beta} + n_1\frac{\partial n_3}{\partial\beta}\right) \\ \tau'_1 &= h_1\left(l_2\frac{\partial l_1}{\partial\alpha} + m_2\frac{\partial m_1}{\partial\alpha} + n_2\frac{\partial n_1}{\partial\alpha}\right), & \tau'_2 &= h_2\left(l_2\frac{\partial l_1}{\partial\beta} + m_2\frac{\partial m_1}{\partial\beta} + n_2\frac{\partial n_1}{\partial\beta}\right) \end{aligned}\right\}. \quad (5)$$

According to the general principles of KIRCHHOFF's method, we may for a first approximation omit the u', v', w' which occur in equations (2) and (3), thus re-writing :—

$$\left.\begin{aligned} \frac{\partial u'}{\partial p} &= \frac{\sigma_1}{h_1} + p\frac{\partial}{\partial\alpha}\left(\frac{1}{h_1}\right) - \frac{\tau'_1}{h_1h_2}q + \frac{\lambda'_1}{h_1h_3}r \\ \frac{\partial v'}{\partial p} &= q\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right) - \frac{\kappa'_1}{h_1h_3}r + \frac{\tau'_1}{h_1^2}p \\ \frac{\partial w'}{\partial p} &= r\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right) - \frac{\lambda'_1}{h_1^2}p + \frac{\kappa'_1}{h_1h_2}q \end{aligned}\right\}, \quad \ldots \ldots (6)$$

$$\left.\begin{aligned} \frac{\partial u'}{\partial q} &= \frac{\varpi}{h_2} + p\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) - \frac{\tau'_2}{h_2^2}q + \frac{\lambda'_2}{h_2h_3}r \\ \frac{\partial v'}{\partial q} &= \frac{\sigma_2}{h_2} + q\frac{\partial}{\partial\beta}\left(\frac{1}{h_2}\right) - \frac{\kappa'_2}{h_2h_3}r + \frac{\tau'_2}{h_2h_1}p \\ \frac{\partial w'}{\partial q} &= r\frac{\partial}{\partial\beta}\left(\frac{1}{h_3}\right) - \frac{\lambda'_2}{h_2h_1}p + \frac{\kappa'_2}{h_2^2}q \end{aligned}\right\}. \quad \ldots \ldots (7)$$

Since we must have

$$\frac{\partial^2}{\partial p\,\partial q}\begin{matrix}u'\\v'\\w'\end{matrix} = \frac{\partial^2}{\partial q\,\partial p}\begin{matrix}u'\\v'\\w'\end{matrix},$$

we find

$$\tau'_1 = -h_1h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right), \quad \tau'_2 = h_1h_2\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right), \quad \lambda'_2 = -\kappa'_1. \quad \ldots (8)$$

Let K_1, Λ_1, T_1, K_2, Λ_2, T_2 be the values of κ'_1, λ'_1, τ'_1, κ'_2, λ'_2, τ'_2 before strain, and let $\kappa'_1 - \mathrm{K}_1 = \kappa_1$, $\lambda'_1 - \Lambda_1 = \lambda_1$, $\tau_1 - \mathrm{T}_1 = \tau_1$, and similarly for the others, then

$$\mathrm{K}_1 = -\Lambda_2, \quad \kappa_1 = -\lambda_2, \quad \tau_1 = \tau_2 = 0.$$

In (6) and (7) suppose $u' = u_0$, $v' = v_0$, $w' = w_0$; then

$$\frac{\partial u_0}{\partial p} = p\frac{\partial}{\partial\alpha}\left(\frac{1}{h_1}\right) + q\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + \frac{\Lambda_1}{h_1 h_3}r, \text{ with five other equations;}$$

subtracting these from (6) and (7), we find

$$\left.\begin{aligned}
\frac{\partial u_1}{\partial p} &= \frac{\sigma_1}{h_1} + \frac{\lambda_1}{h_1 h_3}r, & \frac{\partial u_1}{\partial q} &= \frac{\varpi}{h_2} - \frac{\kappa_1}{h_2 h_3}r,\\
\frac{\partial v_1}{\partial p} &= -\frac{\kappa_1}{h_1 h_3}r, & \frac{\partial v_1}{\partial q} &= \frac{\sigma}{h_2} - \frac{\kappa_2}{h_2 h_3}r,\\
\frac{\partial w_1}{\partial p} &= \frac{\kappa_1}{h_1 h_2}q - \frac{\lambda_1}{h_1^2}p, & \frac{\partial w_1}{\partial q} &= \frac{\kappa_1}{h_1 h_2}p + \frac{\kappa_2}{h_2^2}q.
\end{aligned}\right\} \quad . \quad . \quad . \quad (9)$$

4. These are simply the conditions of continuity of the mass of the shell when deformed. To obtain the forms of u_1, v_1, w_1 from them we shall have to introduce stress-conditions. As the quantities in (9) are small, it will be sufficient to omit products, and so form equations of equilibrium of the element referred to the orthogonal coordinates (p, q, r) as if we were referring to fixed axes at P.

If A, B, C be three functions of r to be determined, we have

$$\left.\begin{aligned}
u_1 &= \mathrm{A} + \frac{\lambda_1}{h_3 h_1}rp - \frac{\kappa_1}{h_2 h_3}qr + \frac{\sigma_1}{h_1}p + \frac{\varpi}{h_2}q,\\
v_1 &= \mathrm{B} - \frac{\kappa_1}{h_3 h_1}rp - \frac{\kappa_2}{h_2 h_3}qr + \frac{\sigma_2}{h_2}q,\\
w_1 &= \mathrm{C} - \tfrac{1}{2}\frac{\lambda_1}{h_1^2}p^2 + \tfrac{1}{2}\frac{\kappa_2}{h_2^2}q^2 + \frac{\kappa_1}{h_1 h_2}pq.
\end{aligned}\right\}$$

Hence, for the six components of strain, and for the cubical dilatation δ,

$$\left.\begin{aligned}
e &= \frac{\lambda_1}{h_3}r + \sigma_1, & f &= -\frac{\kappa_2}{h_3}r + \sigma_2, & g &= h_3\frac{\partial \mathrm{C}}{\partial r},\\
a &= h_3\frac{\partial \mathrm{B}}{\partial r}, & b &= h_3\frac{\partial \mathrm{A}}{\partial r}, & c &= -2\frac{\kappa_1}{h_3}r + \varpi,\\
\delta &= e + f + g = h_3\frac{\partial \mathrm{C}}{\partial r} + \sigma_1 + \sigma_2 - \frac{\kappa_2 - \lambda_1}{h_3}r.
\end{aligned}\right\} \quad . \quad . \quad (10)$$

To determine A, B, C, we have the stress-equations

$$\left.\begin{aligned} h_1 \frac{\partial}{\partial p}(\overline{m-n}\,\delta + 2ne) + h_2 \frac{\partial}{\partial q}(nc) + h_3 \frac{\partial}{\partial r}(nb) = 0, \\ h_1 \frac{\partial}{\partial p}(nc) + h_2 \frac{\partial}{\partial q}(\overline{m-n}\,\delta + 2nf) + h_3 \frac{\partial}{\partial r}(na) = 0, \\ h_1 \frac{\partial}{\partial p}(nb) + h_2 \frac{\partial}{\partial q}(na) + h_3 \frac{\partial}{\partial r}(\overline{m-n}\,\delta + 2ng) = 0, \end{aligned}\right\}$$

where $m = k + \frac{1}{3}n$, k being the modulus of compression, and n that of rigidity.

Hence,

$$\frac{\partial^2 A}{\partial r^2} = 0, \qquad \frac{\partial^2 B}{\partial r^2} = 0,$$

and

$$(m - n)\left(h_3 \frac{\partial^2 C}{\partial r^2} + \frac{\lambda_1 - \kappa_2}{h_3}\right) + 2nh_3 \frac{\partial^2 C}{\partial r^2} = 0.$$

Thus,

$$C = C_1 r + \frac{m-n}{m+n}\,\frac{\kappa_2 - \lambda_1}{h_3^2}\,\frac{r^2}{2}.$$

If there be no surface-tractions on the surfaces initially parallel to the middle-surface, viz., $r = \pm h_3 h$, then $A = 0$, and $B = 0$, and also at the surfaces

$$(m - n)\,\delta + 2ng = 0,$$

so that

$$C_1 = -\frac{\sigma_1 + \sigma_2}{h_3}\,\frac{m-n}{m+n}.$$

Thus,

$$\left.\begin{aligned} u_1 &= \lambda_1\, pr/h_1h_3 - \kappa_1\, qr/h_2h_3 + \sigma_1\, p/h_1 + \varpi\, q/h_2, \\ v_1 &= -\kappa_1\, pr/h_1h_3 - \kappa_2\, qr/h_2h_3 + \sigma_2\, q/h_2, \\ w_1 &= -\tfrac{1}{2}\lambda_1\, p^2/h_1^2 + \tfrac{1}{2}\kappa_2\, q^2/h_2^2 + \kappa_1\, pq/h_1h_2 \\ &\qquad + \frac{m-n}{m+n}\left(\tfrac{1}{2}\,\overline{\kappa_2 - \lambda_1}\, r^2/h_3^2 - \overline{\sigma_1 + \sigma_2}\, r/h_3\right). \end{aligned}\right\} \quad . \quad (11)^*$$

Hence,

* Expressions equivalent to these have been given by ARON, but his work contains an error. His equations (7, a), (7, b), p. 145, are strictly analogous to equations (6) and (7) above, but the terms in $\nu\,\frac{\partial}{\partial\alpha}\left(\frac{1}{h_1}\right)$ . . . are all omitted. The test $\frac{\partial^2}{\partial p\,\partial q}\frac{u'}{v'} = \frac{\partial^2}{\partial q\,\partial p}\frac{u'}{v'}$ is not applied; if it had been, there would have resulted equations which in my notation are $\tau'_1 = 0$, $\tau'_2 = 0$, but the values of τ'_1, τ'_2 are calculated subsequently by the method of Art. 7, and are the same as those given in equations (8).

$$e = \frac{\lambda_1 r}{h_3} + \sigma_1, \qquad f = -\frac{\kappa_2 r}{h_3} + \sigma_2, \qquad g = \frac{m-n}{m+n}\left[\overline{\kappa_2 - \lambda_1}\,\frac{r}{h_3} - \overline{\sigma_1 + \sigma_2}\right],$$

$$a = 0, \qquad b = 0, \qquad c = \varpi - 2\kappa_1 \frac{r}{h_3},$$

$$\delta = \frac{2n}{m+n}\left(\sigma_1 + \sigma_2 - \overline{\kappa_2 - \lambda_1}\,\frac{r}{h_3}\right);$$

and the potential energy per unit volume is

$$\tfrac{1}{2}\left[(m-n)\,\delta^2 + 2n\,(e^2 + f^2 + g^2) + n\,(a^2 + b^2 + c^2)\right]$$
$$= nz^2\left[\kappa_2^2 + \lambda_1^2 + 2\kappa_1^2 + \frac{m-n}{m+n}(\kappa_2 - \lambda_1)^2\right] + n\left[(\sigma_1^2 + \sigma_2^2 + \tfrac{1}{2}\varpi^2) + \frac{m-n}{m+n}(\sigma_1 + \sigma_2)^2\right]$$
$$+ \text{ a term in } z,$$

where z is written for r/h_3.

Multiplying this expression by dz, and integrating from h to $-h$, the term in z disappears, and we find for the potential energy per unit area

$$\mathrm{W} = \tfrac{2}{3}\,nh^3\left[\kappa_2^2 + \lambda_1^2 + 2\kappa_1^2 + \frac{m-n}{m+n}(\kappa_2 - \lambda_1)^2\right]$$
$$+ 2nh\left[\sigma_1^2 + \sigma_2^2 + \tfrac{1}{2}\varpi^2 + \frac{m-n}{m+n}(\sigma_1 + \sigma_2)^2\right],$$

or

$$\left.\begin{aligned} \mathrm{W} = \tfrac{4}{3}\,nh^3\,\frac{m}{m+n}&\left[(\kappa_2 - \lambda_1)^2 + \frac{m+n}{m}(\kappa_2\lambda_1 + \kappa_1^2)\right] \\ &+ 2nh\left[\sigma_1^2 + \sigma_2^2 + \tfrac{1}{2}\varpi^2 + \frac{m-n}{m+n}(\sigma_1 + \sigma_2)^2\right] \end{aligned}\right\}. \qquad (12)$$

The term containing h^3 is the term depending on the bending, and the term containing h is the term depending on the stretching of the middle-surface. We shall hereafter denote by W_1, W_2 the expressions

$$(\kappa_2 - \lambda_1)^2 + \frac{m+n}{m}(\kappa_2\lambda_1 + \kappa_1^2),$$

$$\sigma_1^2 + \sigma_2^2 + \tfrac{1}{2}\varpi^2 + \frac{m-n}{m+n}(\sigma_1 + \sigma_2)^2.$$

§ 4. *Geometrical Theory of Small Deformation of Extensible Surfaces.*

5. We have now, by means of equations (4) and (5), to express the potential energy in terms of the displacement of a point on the middle-surface.

Let u, v, w denote the displacements of the point P on the middle-surface, u being parallel to $\beta =$ const., v to $\alpha =$ const., and w along the normal outwards, and let $\partial/\partial n_3$ denote differentiation along the normal.

The square of the length of a line joining two near points of the surface before strain is

$$\left(\frac{d\alpha}{h_1}\right)^2 + \left(\frac{d\beta}{h_2}\right)^2,$$

and the square of the length of the line joining the same two points after strain is

$$\left[\frac{d\alpha}{h_1}(1+\sigma_1)\right]^2 + \left[\frac{d\beta}{h_2}(1+\sigma_2)\right]^2 + 2\varpi\frac{d\alpha\, d\beta}{h_1 h_2},$$

neglecting small quantities of a higher order. But this same square of the new length of the line is

$$[d\alpha + d(h_1 u)]^2\left[\frac{1}{h_1} + \delta\left(\frac{1}{h_1}\right)\right]^2 + [d\beta + d(h_2 v)]^2\left[\frac{1}{h_2} + \delta\left(\frac{1}{h_2}\right)\right]^2 + (dw)^2,$$

where $\delta(1/h_1)$, $\delta(1/h_2)$ are the increments of $1/h_1$, $1/h_2$ produced by strain, so that

$$\delta\left(\frac{1}{h_1}\right) = h_1 u\frac{\partial}{\partial\alpha}\left(\frac{1}{h_1}\right) + h_2 v\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + w\frac{\partial}{\partial n_3}\left(\frac{1}{h_1}\right),$$

and similarly for h_2, also

$$d(h_1 u) = d\alpha\frac{\partial}{\partial\alpha}(h_1 u) + d\beta\frac{\partial}{\partial\beta}(h_1 u),$$

and so for $d(h_2 v)$.

In the two expressions for the square of the new length of the line we may equate coefficients of $(d\alpha)^2$, $(d\beta)^2$, and $(d\alpha\, d\beta)$, and omit powers of u, v, w, or their differential coefficients above the first; thus

$$\left.\begin{aligned}\sigma_1 &= h_1\frac{\partial u}{\partial\alpha} + h_1 h_2 v\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + \frac{w}{\rho_1},\\ \sigma_2 &= h_2\frac{\partial v}{\partial\beta} + h_1 h_2 u\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right) + \frac{w}{\rho_2},\\ \varpi &= \frac{h_2}{h_1}\frac{\partial}{\partial\beta}(h_1 u) + \frac{h_1}{h_2}\frac{\partial}{\partial\alpha}(h_2 v),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where we have written

$$\frac{1}{\rho_1} = h_1\frac{\partial}{\partial n_3}\left(\frac{1}{h_1}\right), \qquad \frac{1}{\rho_2} = h_2\frac{\partial}{\partial n_3}\left(\frac{1}{h_2}\right),$$

in accordance with LAMÉ's result ('Leçons sur les Coordonnées Curvilignes,' p. 51), viz., ρ_1, ρ_2 are the principal radii of curvature of the middle-surface before strain, reckoned inwards.

6. To express κ_2, λ_1, κ_1 in the same way, we suppose a system of fixed axes at P, whose directions coincide with those of the (x, y, z) axes at P before strain. The coordinates referred to P of a point near P on the deformed surface are

$$\delta\xi = \frac{d\alpha}{h_1} + du - v\,\delta\theta_3 + w\,\delta\theta_2,$$

$$\delta\eta = \frac{d\beta}{h_2} + dv - w\delta\theta_1 + u\,\delta\theta_3,$$

$$\delta\zeta = \quad dw - u\,\delta\theta_2 + v\,\delta\theta_1,$$

where $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$, are the elementary rotations of the axes (x, y, z) about themselves, when the origin is changed from α, β to $\alpha + \delta\alpha$, $\beta + \delta\beta$, viz. :—

$$\left.\begin{aligned} \delta\theta_1 &= -\frac{\partial}{\partial n_3}\left(\frac{1}{h_2}\right)d\beta = -\frac{1}{h_2\rho_2}d\beta, \\ \delta\theta_2 &= \frac{\partial}{\partial n_3}\left(\frac{1}{h_1}\right)d\alpha = \frac{1}{h_1\rho_1}d\alpha, \\ \delta\theta_3 &= h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)d\beta - h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)d\alpha. \end{aligned}\right\} \quad \ldots \ldots \quad (14)$$

Hence, by equations (4),

$$l_1 = (1-\sigma_1)\left[1 + h_1\frac{\partial u}{\partial\alpha} + h_1 h_2 v\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + \frac{w}{\rho_1}\right],$$

$$m_1 = (1-\sigma_1)\left[h_1\frac{\partial v}{\partial\alpha} - h_1 h_2 u\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\right],$$

$$n_1 = (1-\sigma_1)\left[h_1\frac{\partial w}{\partial\alpha} - \frac{u}{\rho_1}\right],$$

$$L_2 = (1-\sigma_2)\left[h_2\frac{\partial u}{\partial\beta} - h_1 h_2 v\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\right],$$

$$M_2 = (1-\sigma_2)\left[1 + h_2\frac{\partial v}{\partial\beta} + h_1 h_2 u\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right) + \frac{w}{\rho_2}\right],$$

$$N_2 = (1-\sigma_2)\left[h_2\frac{\partial w}{\partial\beta} - \frac{v}{\rho_2}\right];$$

substituting for σ_1, σ_2, and neglecting small quantities of the second order, we find

$$\left.\begin{aligned} &l_1 = 1, && m_1 = h_1\frac{\partial v}{\partial\alpha} - h_1 h_2 u\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right), && n_1 = h_1\frac{\partial w}{\partial\alpha} - \frac{u}{\rho_1}, \\ &L_2 = h_2\frac{\partial u}{\partial\beta} - h_1 h_2 v\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right), && M_2 = 1, && N_2 = h_2\frac{\partial w}{\partial\beta} - \frac{v}{\rho_2}, \\ &\text{whence,} && && \\ &l_3 = -h_1\frac{\partial w}{\partial\alpha} + \frac{u}{\rho_1}, && m_3 = -h_2\frac{\partial w}{\partial\beta} + \frac{v}{\rho_2}, && n_3 = 1. \end{aligned}\right\} \quad (15)$$

These are the $l_1, m_1, n_1 \ldots$, referred to axes at each point determined by three certain line-elements at the point, if δ denote the change in the value of these as we go from any point P to a near point on the middle-surface, then referred to the fixed axes at P, we have

$$\delta l_1 = \frac{\partial l_1}{\partial \alpha} d\alpha + \frac{\partial l_1}{\partial \beta} d\beta - m_1 \delta\theta_3 + n_1 \delta\theta_2, \text{ and so on };$$

so that

$$\delta l_1 = d\alpha \left[h_2 \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right) \left\{ h_1 \frac{\partial v}{\partial \alpha} - h_1 h_2 u \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right) \right\} + \frac{1}{h_1 \rho_1}\left(h_1 \frac{\partial w}{\partial \alpha} - \frac{u}{\rho_1} \right) \right]$$
$$- d\beta \left[h_1 \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right) \left\{ h_1 \frac{\partial v}{\partial \alpha} - h_1 h_2 u \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right) \right\} \right],$$

$$\delta m_1 = d\alpha \left[\frac{\partial}{\partial \alpha} \left\{ h_1 \frac{\partial v}{\partial \alpha} - h_1 h_2 u \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right) \right\} - h_2 \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right) \right]$$
$$+ d\beta \left[\frac{\partial}{\partial \beta} \left\{ h_1 \frac{\partial v}{\partial \alpha} - h_1 h_2 u \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right) \right\} + \frac{1}{h_2 \rho_2}\left(h_1 \frac{\partial w}{\partial \alpha} - \frac{u}{\rho_1} \right) + h_1 \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right) \right],$$

$$\delta n_1 = d\alpha \left[\frac{\partial}{\partial \alpha} \left\{ h_1 \frac{\partial w}{\partial \alpha} - \frac{u}{\rho_1} \right\} - \frac{1}{h_1 \rho_1} \right]$$
$$+ d\beta \left[\frac{\partial}{\partial \beta} \left\{ h_1 \frac{\partial w}{\partial \alpha} - \frac{u}{\rho_1} \right\} - \frac{1}{h_2 \rho_2} \left\{ h_1 \frac{\partial v}{\partial \alpha} - h_1 h_2 u \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right) \right\} \right];$$

in the same way

$$\delta L_2 = d\alpha \left[\frac{\partial}{\partial \alpha} \left\{ h_2 \frac{\partial u}{\partial \beta} - h_1 h_2 v \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right) \right\} + h_2 \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right) + \frac{1}{h_1 \rho_1}\left(h_2 \frac{\partial w}{\partial \beta} - \frac{v}{\rho_2} \right) \right]$$
$$+ d\beta \left[\frac{\partial}{\partial \beta} \left\{ h_2 \frac{\partial u}{\partial \beta} - h_1 h_2 v \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right) \right\} - h_1 \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right) \right],$$

$$\delta M_2 = - d\alpha \left[h_2 \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right) \left\{ h_2 \frac{\partial u}{\partial \beta} - h_1 h_2 v \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right) \right\} \right]$$
$$+ d\beta \left[\frac{1}{h_2 \rho_2}\left(h_2 \frac{\partial w}{\partial \beta} - \frac{v}{\rho_2} \right) + h_1 \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right) \left\{ h_2 \frac{\partial u}{\partial \beta} - h_1 h_2 v \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right) \right\} \right],$$

$$\delta N_2 = d\alpha \left[\frac{\partial}{\partial \alpha} \left\{ h_2 \frac{\partial w}{\partial \beta} - \frac{v}{\rho_2} \right\} - \frac{1}{h_1 \rho_1} \left\{ h_2 \frac{\partial u}{\partial \beta} - h_1 h_2 v \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right) \right\} \right]$$
$$+ d\beta \left[\frac{\partial}{\partial \beta} \left\{ h_2 \frac{\partial w}{\partial \beta} - \frac{v}{\rho_2} \right\} - \frac{1}{h_2 \rho_2} \right].$$

We may form the $\kappa'_2, \lambda'_1, \kappa'_1$ from these, for $m_1 \varpi$, $n_1 \varpi$ are small quantities of the second order, and $l_3 \dfrac{\partial (l_1 \varpi)}{\partial \alpha}$, $l_3 \dfrac{\partial (l_1 \varpi)}{\partial \beta}$ are also of the second order; hence, to the first order, using equations (5) we obtain

$$\left.\begin{aligned}
\kappa_1 &= h_1\left[-\left(h_1\frac{\partial w}{\partial\alpha}-\frac{u}{\rho_1}\right)h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)+\frac{\partial}{\partial\alpha}\left(h_2\frac{\partial w}{\partial\beta}-\frac{v}{\rho_2}\right)\right.\\
&\qquad\left.-\frac{1}{h_1\rho_1}\left\{h_2\frac{\partial u}{\partial\beta}-h_1h_2v\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\right\}\right],\\
\lambda'_1 &= -h_1\left[-\left(h_2\frac{\partial w}{\partial\beta}-\frac{v}{\rho_2}\right)\left\{-h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\right\}+\frac{\partial}{\partial\alpha}\left(h_1\frac{\partial w}{\partial\alpha}-\frac{u}{\rho_1}\right)-\frac{1}{h_1\rho_1}\right],\\
\kappa'_2 &= h_2\left[-\left(h_1\frac{\partial w}{\partial\alpha}-\frac{u}{\rho_1}\right)\left\{-h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\right\}+\frac{\partial}{\partial\beta}\left(h_2\frac{\partial w}{\partial\beta}-\frac{v}{\rho_2}\right)-\frac{1}{h_2\rho_2}\right],\\
\lambda'_2 &= -h_2\left[-\left(h_2\frac{\partial w}{\partial\beta}-\frac{v}{\rho_2}\right)h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)+\frac{\partial}{\partial\beta}\left(h_1\frac{\partial w}{\partial\alpha}-\frac{u}{\rho_1}\right)\right.\\
&\qquad\left.-\frac{1}{h_2\rho_2}\left\{h_1\frac{\partial v}{\partial\alpha}-h_1h_2u\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\right\}\right].
\end{aligned}\right\}\quad(16)$$

The relation $\lambda'_2 = -\kappa'_1$ reduces to

$$u\left[\frac{\partial}{\partial\beta}\left(\frac{1}{h_1\rho_1}\right)-\frac{1}{\rho_2}\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\right]=v\left[\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2\rho_2}\right)-\frac{1}{\rho_1}\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\right],\quad . \quad . \quad . \quad (17)$$

and each of these expressions vanishes (LAMÉ, p. 80); thus, this condition is fulfilled identically.* Using these relations, we find

$$\left.\begin{aligned}
\kappa_1 &= h_1h_2\frac{\partial^2 w}{\partial\alpha\,\partial\beta}+h_1\frac{\partial h_2}{\partial\alpha}\frac{\partial w}{\partial\beta}+h_2\frac{\partial h_1}{\partial\beta}\frac{\partial w}{\partial\alpha}-\frac{h_1}{\rho_2}\frac{\partial v}{\partial\alpha}-\frac{h_2}{\rho_1}\frac{\partial u}{\partial\beta}\\
&\qquad+h_1h_2\left\{\frac{u}{\rho_1}\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)+\frac{v}{\rho_2}\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\right\},\\
-\lambda_1 &= h_1^2\frac{\partial^2 w}{\partial\alpha^2}+h_1\frac{\partial h_1}{\partial\alpha}\frac{\partial w}{\partial\alpha}+h_2^2h_1\frac{\partial w}{\partial\beta}\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)-\frac{h_1}{\rho_1}\frac{\partial u}{\partial\alpha}\\
&\qquad-h_1u\frac{\partial}{\partial\alpha}\left(\frac{1}{\rho_1}\right)-h_1h_2v\frac{1}{\rho_2}\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right),\\
\kappa_2 &= h_2^2\frac{\partial^2 w}{\partial\beta^2}+h_2\frac{\partial h_2}{\partial\beta}\frac{\partial w}{\partial\beta}+h_1^2h_2\frac{\partial w}{\partial\alpha}\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)-\frac{h_2}{\rho_2}\frac{\partial v}{\partial\beta}\\
&\qquad-h_2v\frac{\partial}{\partial\beta}\left(\frac{1}{\rho_2}\right)-h_1h_2u\frac{1}{\rho_1}\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right).
\end{aligned}\right\}\quad(18)$$

7. The quantities defined by equations (5) have been calculated directly; we wish to obtain an interpretation in terms of quantities defining the curvature of the middle-surface after strain.

* This may be taken as a verification in some degree of the preceding work. In endeavouring to form equations referred to the above set of moving axes, ARON neglects the $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$ and deduces values of λ'_2, κ'_1 (my notation), which do not satisfy the relation $\lambda'_2+\kappa'_1=0$ (see the memoir above quoted, pp. 169 *et seq.*). In consequence, he is obliged to make an assumption that $\partial(vh_2^{-1})/\partial\alpha$ is a small quantity of the second order.

If the relations (17) had not been known, the theory of deformation would prove them.

First, suppose we are dealing with an inextensible surface, then

$$l_1 = h_1 \frac{\partial \xi}{\partial \alpha}, \qquad m_1 = h_1 \frac{\partial \eta}{\partial \alpha}, \qquad n_1 = h_1 \frac{\partial \zeta}{\partial \alpha},$$

$$l_2 = h_2 \frac{\partial \xi}{\partial \beta}, \qquad m_2 = h_2 \frac{\partial \eta}{\partial \beta}, \qquad n_2 = h_2 \frac{\partial \zeta}{\partial \beta}.$$

By equations (5), since $l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$, and $l_2 l_3 + m_2 m_3 + n_2 n_3 = 0$,

$$-\lambda'_1 = h_1^3 h_2 \left[\frac{\partial(\eta\zeta)}{\partial(\alpha\beta)} \frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial(\zeta\xi)}{\partial(\alpha\beta)} \frac{\partial^2 \eta}{\partial \alpha^2} + \frac{\partial(\xi\eta)}{\partial(\alpha\beta)} \frac{\partial^2 \zeta}{\partial \alpha^2}\right],$$

$$\kappa'_1 = h_1^2 h_2^2 \left[\frac{\partial(\eta\zeta)}{\partial(\alpha\beta)} \frac{\partial^2 \xi}{\partial \alpha\, \partial \beta} + \frac{\partial(\zeta\xi)}{\partial(\alpha\beta)} \frac{\partial^2 \eta}{\partial \alpha\, \partial \beta} + \frac{\partial(\xi\eta)}{\partial(\alpha\beta)} \frac{\partial^2 \zeta}{\partial \alpha\, \partial \beta}\right],$$

$$-\lambda'_2 = h_1^2 h_2^2 \left[\frac{\partial(\eta\zeta)}{\partial(\alpha\beta)} \frac{\partial^2 \xi}{\partial \alpha\, \partial \beta} + \frac{\partial(\zeta\xi)}{\partial(\alpha\beta)} \frac{\partial^2 \eta}{\partial \alpha\, \partial \beta} + \frac{\partial(\xi\eta)}{\partial(\alpha\beta)} \frac{\partial^2 \zeta}{\partial \alpha\, \partial \beta}\right],$$

$$\kappa'_2 = h_1 h_2^3 \left[\frac{\partial(\eta\zeta)}{\partial(\alpha\beta)} \frac{\partial^2 \xi}{\partial \beta^2} + \frac{\partial(\zeta\xi)}{\partial(\alpha\beta)} \frac{\partial^2 \eta}{\partial \beta^2} + \frac{\partial(\xi\eta)}{\partial(\alpha\beta)} \frac{\partial^2 \zeta}{\partial \beta^2}\right].$$

Hence, taking the notation of SALMON's 'Geometry of Three Dimensions,' chapter 12, section 4, we have

$$\mathrm{E} = \frac{1}{h_1^2}, \qquad \mathrm{F} = 0, \qquad \mathrm{G} = \frac{1}{h_2^2}, \qquad \mathrm{V} = \frac{1}{h_1 h_2},$$

$$h_1^2 h_2^2 \mathrm{F}' = \kappa'_1, \qquad h_1 h_2^3 \mathrm{G} = \kappa'_2, \qquad h_1^3 h_2 \mathrm{E}' = -\lambda'_1;$$

and the equation for the principal radii of curvature is*

$$\left(\frac{\lambda'_1}{h_1^2}\rho - \frac{1}{h_1^2}\right)\left(-\frac{\kappa'_2}{h_2^2}\rho - \frac{1}{h_2^2}\right) - \frac{\kappa'^2_1}{h_1^2 h_2^2}\rho^2 = 0,$$

so that, if ρ'_1, ρ'_2 be the roots of this equation,

$$\kappa'_2 - \lambda'_1 = -\frac{1}{\rho'_1} - \frac{1}{\rho'_2},$$

$$\kappa'_2 \lambda'_1 + \kappa'^2_1 = -\frac{1}{\rho'_1 \rho'_2}.$$

Also

$$\kappa_2 \lambda_1 + \kappa_1^2 = \kappa'_2 \lambda'_1 + \kappa'^2_1 + \mathrm{K}_2 \Lambda_1 - \Lambda_1 \kappa'_2 - \mathrm{K}_2 \lambda'_1$$
$$= -\frac{1}{\rho'_1 \rho'_2} - \frac{1}{\rho_1 \rho_2} - \frac{\kappa'_2}{\rho_1} + \frac{\lambda'_1}{\rho_2}.$$

* SALMON, p. 346. I have changed the sign of ρ so that the roots shall be the ρ_1 and ρ_2 of Art. 5.

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In the case of the sphere, this is

$$-\frac{1}{\rho'_1\rho'_2}-\frac{1}{a^2}+\frac{1}{a}\left(\frac{1}{\rho'_1}+\frac{1}{\rho'_2}\right),\quad \text{where } a \text{ is the radius,}$$

or

$$-\delta\left(\frac{1}{\rho_1}\right)\delta\left(\frac{1}{\rho_2}\right);$$

for any other inextensible surface this will not be the case.

Now, suppose the surface extensible, and consider (α, β) as two parameters defining a point on the deformed surface; in this view they will not be orthogonal parameters, and we find

$$\frac{1}{h_1h_2}\varpi(1+\sigma_1)(1+\sigma_2)=\frac{\partial\xi}{\partial\alpha}\frac{\partial\xi}{\partial\beta}+\frac{\partial\eta}{\partial\alpha}\frac{\partial\eta}{\partial\beta}+\frac{\partial\zeta}{\partial\alpha}\frac{\partial\zeta}{\partial\beta}=\mathrm{F},$$

or

$$\mathrm{F}=\frac{\varpi}{h_1h_2}\ \text{to the first order};$$

so

$$\mathrm{E}=\frac{1-2\sigma_1}{h_1^2},\qquad \mathrm{G}=\frac{1-2\sigma_2}{h_2^2}.$$

Again,

$$l_1=\frac{h_1}{1+\sigma_1}\frac{\partial\xi}{\partial\alpha},$$

with similar expressions for m_1, n_1.

To find κ'_2, λ'_1, κ'_1 from the definitions in equations (5), we notice that the terms in

$$\frac{\partial}{\partial\alpha}\left(\frac{h_1}{1+\sigma_1}\right),\qquad \frac{\partial}{\partial\beta}\left(\frac{h_1}{1+\sigma_1}\right)$$

will always be multiplied by terms of the form

$$l_3\frac{\partial\xi}{\partial\alpha}+m_3\frac{\partial\eta}{\partial\alpha}+n_3\frac{\partial\zeta}{\partial\alpha}.$$

Now

$$l_3=(m_1n_2-m_2n_1)=m_1\mathrm{N}_2-n_1\mathrm{M}_2=\frac{h_1}{1+\sigma_1}\frac{h_2}{1+\sigma_2}\frac{\partial(\eta\zeta)}{\partial(\alpha\beta)},$$

and similarly for m_3, n_3; thus, the differential coefficients of $h_1/(1+\sigma_1)$, $h_2/(1+\sigma_2)$ will be multiplied by factors which vanish identically, viz., they are of the form

$$\frac{\partial\xi}{\partial\alpha}\frac{\partial(\eta\zeta)}{\partial(\alpha\beta)}+\frac{\partial\eta}{\partial\alpha}\frac{\partial(\zeta\xi)}{\partial(\alpha\beta)}+\frac{\partial\zeta}{\partial\alpha}\frac{\partial(\xi\eta)}{\partial(\alpha\beta)}.$$

Hence,

$$\mathrm{E}'=-\frac{(1+\sigma_1)\lambda'_1}{h_1^3h_2},\qquad \mathrm{F}'=\frac{\kappa_1}{h_1^2h_2^2},\qquad \mathrm{G}'=\frac{(1+\sigma_2)\kappa'_2}{h_2^3h_1}.$$

The equation for the radii of curvature being

$$(E'\rho + EV)(G'\rho + GV) - (F'\rho + FV)^2 = 0,$$

where

$$V = \sqrt{(EG - F^2)} = \frac{1 - \sigma_1 - \sigma_2}{h_1 h_2} \text{ nearly,}$$

we find that to the first order

$$\frac{1}{\rho'_1} = \lambda'_1(1 + \sigma_2 + 4\sigma_1) = \frac{1}{\rho_1} + \frac{\sigma_2 + 4\sigma_1}{\rho_1} + \lambda_1,$$

$$\frac{1}{\rho'_2} = -\kappa'_2(1 + \sigma_1 + 4\sigma_2) = \frac{1}{\rho_2} + \frac{\sigma_1 + 4\sigma_2}{\rho_2} - \kappa_2.$$

We have already found expressions for κ_2, λ_1, κ_1 in terms of the displacements; hence, we have found expressions for the new principal curvatures and the position of the new principal planes, in terms of the displacements, for the position of these planes depends only on F' or on κ_1; we have also found the interpretation of the κ_2, λ_1, κ_1 in terms of the quantities defining the curvature and the extension.

In the case of an inextensible sphere, the potential energy due to bending is

$$\tfrac{4}{3} nh^3 \frac{m}{m+n} \left[\left\{ \delta\left(\frac{1}{\rho_1}\right) + \delta\left(\frac{1}{\rho_2}\right) \right\}^2 - \frac{m+n}{m} \delta\left(\frac{1}{\rho_1}\right) \delta\left(\frac{1}{\rho_2}\right) \right].$$

For any other surface, whether extensible or not, this will not be the case. If the middle-surface were unextended, the above would be right to small quantities of the first order, but we always require the potential energy correct to small quantities of the second order.

§ 5. *Equations of Motion and Boundary-Conditions.*

8. Following KIRCHHOFF's method, we are going to apply the principle of virtual work to obtain the differential equations of motion and equilibrium, and the boundary-conditions.

Let X_1, Y_1, Z_1 be the components of the bodily force per unit mass parallel to the lines of curvature $\beta =$ const., $\alpha =$ const., and perpendicular to the tangent plane to the middle-surface, acting at any point Q of the shell. Let QP be perpendicular to the middle-surface before strain, and let l_3, m_3, n_3 be the direction cosines of QP after strain referred to axes at P, as in Artt. 1, 6; if u, v, w be the displacements of P, and z the distance PQ, then, when a small variation in the configuration is made, the displacements of Q will be found from equations (1), dropping the p, q, to be

$$\delta u + z\,\delta l_3$$
$$\delta v + z\,\delta m_3$$
$$\delta w + z\,\delta n_3.$$

Let A_1, B_1, C_1 be the components of the system of forces per unit area applied to the edge of the shell, and holding it in its actual configuration. The systems of forces X_1, Y_1, Z_1 acting at all points of a line through P perpendicular to the middle-surface, and the similar (A_1, B_1, C_1) system, will each reduce to a resultant force and couple.

The resultant of the X_1, Y_1, Z_1 system is a force at P whose components are

$$X = \int_{-h}^{h} X_1\,dz, \quad Y = \int_{-h}^{h} Y_1\,dz, \quad Z = \int_{-h}^{h} Z_1\,dz \quad \text{per unit area,}$$

and a couple whose components are

$$L = -\int_{-h}^{h} Y_1\,z\,dz, \quad M = +\int_{-h}^{h} X_1\,z\,dz, \quad 0 \quad \text{per unit area.}$$

The resultant of the A_1, B_1, C_1 system is a force at the point P in which the middle-surface cuts the edge, whose components are

$$A = \int_{-h}^{h} A_1\,dz, \quad B = \int_{-h}^{h} B_1\,dz, \quad C = \int_{-h}^{h} C_1\,dz \quad \text{per unit length of the curve in}$$

which the middle-surface cuts the edge, and a couple whose components are

$$U = -\int_{-h}^{h} B_1\,z\,dz, \quad V = +\int_{-h}^{h} A_1\,z\,dz, \quad 0 \quad \text{per unit length of the same curve.}$$

The general variational equation of motion is

$$\begin{aligned}
0 = &-\iiint [X_1(\delta u + z\,\delta l_3) + Y_1(\delta v + z\,\delta m_3) + Z_1(\delta w + z\,\delta n_3)]\,dS\,dz \\
&-\iint [A_1(\delta u + z\,\delta l_3) + B_1(\delta v + z\,\delta m_3) + C_1(\delta w + z\,\delta n_3)]\,ds\,dz \\
&+\tfrac{4}{3}nh^3\frac{m}{m+n}\iint \delta W_1\,\delta S + 2nh\iint \delta W_2\,dS \\
&+\rho\iiint\Big[\Big(\frac{\partial^2 u}{\partial t^2} + z\frac{\partial^2 l_3}{\partial t^2}\Big)(\delta u + z\,\delta l_3) + \Big(\frac{\partial^2 v}{\partial t^2} + z\frac{\partial^2 m_3}{\partial t^2}\Big)(\delta v + z\,\delta m_3) \\
&\qquad\qquad + \Big(\frac{\partial^2 w}{\partial t^2} + z\frac{\partial^2 n_3}{\partial t^2}\Big)(\delta w + z\,\delta n_3)\Big]\,dS\,dz,
\end{aligned}$$

where $d\mathrm{S}$ is an element of area of the middle-surface and ds an element of arc of the edge.

Observing that by equations (15) $\delta n_3 = 0$, and integrating with respect to z from h to $-h$, we get the equation

$$
\begin{aligned}
&-\iint(\mathrm{X}\,\delta u + \mathrm{Y}\,\delta v + \mathrm{Z}\,\delta w)\,d\mathrm{S} - \iint(\mathrm{M}\,\delta l_3 - \mathrm{L}\,\delta m_3)\,d\mathrm{S} \\
&-\int(\mathrm{A}\,\delta u + \mathrm{B}\,\delta v + \mathrm{C}\,\delta w)\,ds - \int(\mathrm{V}\,\delta l_3 - \mathrm{U}\,\delta m_3)\,ds \\
&+\tfrac{4}{3}nh^3\frac{m}{m+n}\iint\delta\mathrm{W}_1\,d\mathrm{S} + 2nh\iint\delta\mathrm{W}_2\,d\mathrm{S} \\
&+2\rho h\iint\left(\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v + \frac{\partial^2 w}{\partial t^2}\delta w\right)d\mathrm{S} + \tfrac{2}{3}\rho h^3\iint\left(\frac{\partial^2 l_3}{\partial t^2}\delta l_3 + \frac{\partial^2 m_3}{\partial t^2}\delta m_3\right)d\mathrm{S} = 0, \quad (19)
\end{aligned}
$$

which contains in itself all the equations and conditions of the problem.

All the double integrals which occur in this equation can be expressed, partly as surface-integrals over the middle-surface, and partly as line-integrals round the edge, by means of the theorem,

$$\iint\left(\frac{d\mathrm{X}}{d\alpha} + \frac{d\mathrm{Y}}{d\beta}\right)d\alpha\,d\beta = \int(\mathrm{X}\lambda h_2 + \mathrm{Y}\mu h_1)\,ds, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where the first integration extends to all values of (α, β) which correspond to points on a surface having s for an edge, and λ, μ are the cosines of the angles which the normal to the edge drawn on the surface and produced outwards makes with the directions of the lines $\beta = \text{const.}$, $\alpha = \text{const.}$ at the edge.

To prove this theorem,* let a line of curvature $\alpha = \text{const.}$ meet the edge in an even number of points, and let $\mathrm{X}_1, \mathrm{X}_2, \ldots$ be the values of X at these points, then

$$\int \mathrm{X}\lambda h_2\,ds = \int[(\mathrm{X}_2 - \mathrm{X}_1) + \ldots]\,h_2\frac{d\beta}{h_2} = \int\{[\mathrm{X}_2 - \mathrm{X}_1] + \ldots\}\,d\beta = \iint\frac{\partial\mathrm{X}}{\partial\alpha}\,d\alpha\,d\beta;$$

so

$$\int \mathrm{Y}\mu h_1\,ds = \iint\frac{\partial\mathrm{Y}}{\partial\beta}\,d\alpha\,d\beta.$$

The partial integrations will be effected by means of the relations

$$
\left.
\begin{aligned}
\mathrm{X}\frac{\partial^2\delta\phi}{\partial\alpha^2} &= \frac{\partial^2\mathrm{X}}{\partial\alpha^2}\delta\phi + \frac{\partial}{\partial\alpha}\left(\mathrm{X}\frac{\partial\,\delta\phi}{\partial\alpha} - \frac{\partial\mathrm{X}}{\partial\alpha}\delta\phi\right), \\
2\mathrm{X}\frac{\partial^2\delta\phi}{\partial\alpha\,\partial\beta} &= 2\frac{\partial^2\mathrm{X}}{\partial\alpha\,\partial\beta}\delta\phi + \frac{\partial}{\partial\alpha}\left(\mathrm{X}\frac{\partial\,\delta\phi}{\partial\beta} - \frac{\partial\mathrm{X}}{\partial\beta}\delta\phi\right) + \frac{\partial}{\partial\beta}\left(\mathrm{X}\frac{\partial\,\delta\phi}{\partial\alpha} - \frac{\partial\mathrm{X}}{\partial\alpha}\delta\phi\right), \\
\mathrm{X}\frac{\partial\,\delta\phi}{\partial\alpha} &= -\frac{\partial\mathrm{X}}{\partial\alpha}\delta\phi + \frac{\partial}{\partial\alpha}(\mathrm{X}\delta\phi).
\end{aligned}
\right\} \quad . \quad (21)
$$

* *Cf.* MAXWELL, 'Electricity and Magnetism,' Art. 21. This theorem is otherwise proved by ARON.

In evaluating the line-integrals we shall use the formulæ—

$$\left.\begin{array}{ll} h_1 \dfrac{\partial}{\partial \alpha} = -\mu \dfrac{\partial}{\partial s} + \lambda \dfrac{\partial}{\partial \nu}, & \dfrac{\partial}{\partial s} = -h_1\mu \dfrac{\partial}{\partial \alpha} + h_2\lambda \dfrac{\partial}{\partial \beta}, \\ h_2 \dfrac{\partial}{\partial \beta} = \quad \mu \dfrac{\partial}{\partial \nu} + \lambda \dfrac{\partial}{\partial s}, & \dfrac{\partial}{\partial \nu} = h_1\lambda \dfrac{\partial}{\partial \alpha} + h_2\mu \dfrac{\partial}{\partial \beta}, \end{array}\right\} \quad . \quad . \quad . \quad . \quad (22)$$

in which $d\nu$ is the element of the normal to the edge drawn as above stated.

9. From equations (15) we have—

$$\left.\begin{array}{l} \delta l_3 = -h_1 \dfrac{\partial\, \delta w}{\partial \alpha} + \dfrac{\delta u}{\rho_1} \\ \delta m_3 = -h_2 \dfrac{\partial\, \delta w}{\partial \beta} + \dfrac{\delta v}{\rho_2} \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

so that

$$\iint (\mathrm{M}\, \delta l_3 - \mathrm{L}\, \delta m_3)\, d\mathrm{S} = \iint \left[\mathrm{M}\left(-h_1 \frac{\partial\, \delta w}{\partial \alpha} + \frac{\delta u}{\rho_1}\right) - \mathrm{L}\left(-h_2 \frac{\partial\, \delta w}{\partial \beta} + \frac{\delta v}{\rho_2}\right)\right] \frac{d\alpha\, d\beta}{h_1 h_2}$$

$$= \iint \left[\frac{\mathrm{M}}{h_1 h_2 \rho_1}\, \delta u - \frac{\mathrm{L}}{h_1 h_2 \rho_2}\, \delta v + \left\{\frac{\partial}{\partial \alpha}\left(\frac{\mathrm{M}}{h_2}\right) - \frac{\partial}{\partial \beta}\left(\frac{\mathrm{L}}{h_1}\right)\right\} \delta w\right] d\alpha\, d\beta$$

$$- \iint \left[\frac{\partial}{\partial \alpha}\left(\frac{\mathrm{M}}{h_2}\, \delta w\right) - \frac{\partial}{\partial \beta}\left(\frac{\mathrm{L}}{h_1}\, \delta w\right)\right] d\alpha\, d\beta$$

$$= \iint \left\{\frac{\mathrm{M}}{h_1 h_2 \rho_1}\, \delta u - \frac{\mathrm{L}}{h_1 h_2 \rho_2}\, \delta v + \left\{\frac{\partial}{\partial \alpha}\left(\frac{\mathrm{M}}{h_2}\right) - \frac{\partial}{\partial \beta}\left(\frac{\mathrm{L}}{h_1}\right)\right\} \delta w\right\} d\alpha\, d\beta$$

$$+ \int (\mathrm{L}\mu - \mathrm{M}\lambda)\, \delta w\, ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Again,

$$\int (\mathrm{V}\, \delta l_3 - \mathrm{U}\, \delta m_3)\, ds = \int \left[\mathrm{V}\left(-h_1 \frac{\partial\, \delta w}{\partial \alpha} + \frac{\delta u}{\rho_1}\right) - \mathrm{U}\left(-h_2 \frac{\partial\, \delta w}{\partial \beta} + \frac{\delta v}{\rho_2}\right)\right] ds$$

$$= \int \left\{\frac{\mathrm{V}}{\rho_1}\, \delta u - \frac{\mathrm{U}}{\rho_2}\, \delta v - \mathrm{V}\left(\lambda \frac{\partial\, \delta w}{\partial \nu} - \mu \frac{\partial\, \delta w}{\partial s}\right) + \mathrm{U}\left(\mu \frac{\partial\, \delta w}{\partial \nu} + \lambda \frac{\partial\, \delta w}{\partial s}\right)\right\} ds$$

$$= \int \left[\frac{\mathrm{V}}{\rho_1}\, \delta u - \frac{\mathrm{U}}{\rho_2}\, \delta v - \frac{\partial}{\partial s}(\lambda \mathrm{U} + \mu \mathrm{V})\, \delta w - (\lambda \mathrm{V} - \mu \mathrm{U})\, \delta \frac{\partial w}{\partial \nu}\right] ds, \quad . \quad . \quad . \quad . \quad (25)$$

by integration by parts.

Again,

$$\iint \delta \mathrm{W}_1\, d\mathrm{S}$$

$$= \iint \left(2\kappa_2 - \frac{m-n}{m}\lambda_1\right) \delta\kappa_2 \frac{d\alpha\, d\beta}{h_1 h_2} + \iint \left(2\lambda_1 - \frac{m-n}{m}\kappa_2\right) \delta\lambda_1 \frac{d\alpha\, d\beta}{h_1 h_2}$$

$$+ 2\, \frac{m+n}{m} \iint \kappa_1\, \delta\kappa_1 \frac{d\alpha\, d\beta}{h_1 h_2}.$$

Thus,

$$
\begin{aligned}
&\iint \delta W_1 \, dS \\
= &\iint d\alpha \, d\beta \, \delta w \Bigg[\frac{\partial^2}{\partial\beta^2} \left\{ \frac{h_2}{h_1} \left(2\kappa_2 - \frac{m-n}{m} \lambda_1 \right) \right\} - \frac{\partial^2}{\partial\alpha^2} \left\{ \frac{h_1}{h_2} \left(2\lambda_1 - \frac{m-n}{m} \kappa_2 \right) \right\} \\
&\quad - \frac{\partial}{\partial\beta} \left\{ \frac{1}{h_1} \frac{\partial h_2}{\partial\beta} \left(2\kappa_2 - \frac{m-n}{m} \lambda_1 \right) \right\} + \frac{\partial}{\partial\alpha} \left\{ \frac{1}{h_2} \frac{\partial h_1}{\partial\alpha} \left(2\lambda_1 - \frac{m-n}{m} \kappa_2 \right) \right\} \\
&\quad - \frac{\partial}{\partial\alpha} \left\{ h_1 \frac{\partial}{\partial\alpha} \left(\frac{1}{h_2} \right) \left(2\kappa_2 - \frac{m-n}{m} \lambda_1 \right) \right\} + \frac{\partial}{\partial\beta} \left\{ h_2 \frac{\partial}{\partial\beta} \left(\frac{1}{h_1} \right) \left(2\lambda_1 - \frac{m-n}{m} \kappa_2 \right) \right\} \\
&\quad + 2 \frac{m+n}{m} \left\{ \frac{\partial^2 \kappa_1}{\partial\alpha \, \partial\beta} - \frac{\partial}{\partial\beta} \left(\frac{\kappa_1}{h_2} \frac{\partial h_2}{\partial\alpha} \right) - \frac{\partial}{\partial\alpha} \left(\frac{\kappa_1}{h_1} \frac{\partial h_1}{\partial\beta} \right) \right\} \Bigg] \\
- &\iint d\alpha \, d\beta \, \delta u \Bigg[\frac{1}{\rho_1} \frac{\partial}{\partial\alpha} \left(\frac{1}{h_2} \right) \left(2\kappa_2 - \frac{m-n}{m} \lambda_1 \right) + \frac{1}{\rho_1} \frac{\partial}{\partial\alpha} \left\{ \frac{1}{h_2} \left(2\lambda_1 - \frac{m-n}{m} \kappa_2 \right) \right\} \\
&\quad - 2 \frac{m+n}{m} \left\{ \frac{\partial}{\partial\beta} \left(\frac{\kappa_1}{h_1 \rho_1} \right) + \frac{\kappa_1}{\rho_1} \frac{\partial}{\partial\beta} \left(\frac{1}{h_1} \right) \right\} \Bigg] \\
+ &\iint d\alpha \, d\beta \, \delta v \Bigg[\frac{1}{\rho_2} \frac{\partial}{\partial\beta} \left(\frac{1}{h_1} \right) \left(2\lambda_1 - \frac{m-n}{m} \kappa_2 \right) + \frac{1}{\rho_2} \frac{\partial}{\partial\beta} \left\{ \frac{1}{h_1} \left(2\kappa_2 - \frac{m-n}{m} \lambda_1 \right) \right\} \\
&\quad + 2 \frac{m+n}{m} \left\{ \frac{\partial}{\partial\alpha} \left(\frac{\kappa_1}{h_2 \rho_2} \right) + \frac{\kappa_1}{\rho_2} \frac{\partial}{\partial\alpha} \left(\frac{1}{h_2} \right) \right\} \Bigg] \\
+ &\int \mu h_1 \, ds \Bigg[\frac{h_2}{h_1} \left(2\kappa_2 - \frac{m-n}{m} \lambda_1 \right) \frac{\partial \, \delta w}{\partial\beta} - \frac{\partial}{\partial\beta} \left\{ \frac{h_2}{h_1} \left(2\kappa_2 - \frac{m-n}{m} \lambda_1 \right) \right\} \delta w \\
&\quad + \frac{1}{h_1} \frac{\partial h_2}{\partial\beta} \left(2\kappa_2 - \frac{m-n}{m} \lambda_1 \right) \delta w - h_2 \frac{\partial}{\partial\beta} \left(\frac{1}{h_1} \right) \left(2\lambda_1 - \frac{m-n}{m} \kappa_2 \right) \delta w \\
&\quad + \frac{m+n}{m} \left\{ \kappa_1 \frac{\partial \, \delta w}{\partial\alpha} - \frac{\partial \kappa_1}{\partial\alpha} \delta w + \frac{2}{h_2} \frac{\partial h_2}{\partial\alpha} \kappa_1 \, \delta w \right\} \Bigg] \\
+ &\int \lambda h_2 \, ds \Bigg[- \frac{h_1}{h_2} \left(2\lambda_1 - \frac{m-n}{m} \kappa_2 \right) \frac{\partial \, \delta w}{\partial\alpha} - \frac{\partial}{\partial\alpha} \left\{ \frac{h_1}{h_2} \left(2\lambda_1 - \frac{m-n}{m} \kappa_2 \right) \right\} \delta w \\
&\quad + h_1 \frac{\partial}{\partial\alpha} \left(\frac{1}{h_2} \right) \left(2\kappa_2 - \frac{m-n}{m} \lambda_1 \right) \delta w - \frac{1}{h_2} \frac{\partial h_1}{\partial\alpha} \left(2\lambda_1 - \frac{m-n}{m} \kappa_2 \right) \delta w \\
&\quad + \frac{m+n}{m} \left\{ \kappa_1 \frac{\partial \, \delta w}{\partial\beta} - \frac{\partial \kappa_1}{\partial\beta} \delta w + \frac{2}{h_1} \frac{\partial h_1}{\partial\beta} \kappa_1 \, \delta w \right\} \Bigg] \\
+ &\int ds \left[- \mu h_1 \, \delta v \frac{1}{h_1 \rho_2} \left(2\kappa_2 - \frac{m-n}{m} \lambda_1 \right) + \lambda h_2 \, \delta u \frac{1}{h_2 \rho_1} \left(2\lambda_1 - \frac{m-n}{m} \kappa_2 \right) \right] \\
&\int ds \, 2 \frac{m+n}{m} \left[\mu h_1 \, \delta u \frac{\kappa_1}{h_1 \rho_1} + \lambda h_2 \, \delta v \frac{\kappa_1}{h_2 \rho_2} \right]. \qquad \qquad (26)
\end{aligned}
$$

Using (22), we find for the line-integral part,

$$\int ds\,\frac{\partial\,\delta w}{\partial\nu}\left[\mu^2\left(2\kappa_2-\frac{m-n}{m}\lambda_1\right)-\lambda^2\left(2\lambda_1-\frac{m-n}{m}\kappa_2\right)+2\lambda\,\mu\,\frac{m+n}{m}\kappa_1\right]$$
$$+\int ds\,\frac{\partial\,\delta w}{\partial s}\left[\frac{m+n}{m}\left\{\lambda\mu\,(\kappa_2+\lambda_1)+(\lambda^2-\mu^2)\,\kappa_1\right\}\right]$$
$$+\int ds\,\delta w\,\mu h_1\left[-\frac{\partial}{\partial\beta}\left\{\frac{h_2}{h_1}\left(2\,\kappa_2-\frac{m-n}{m}\lambda_1\right)\right\}+\frac{1}{h_1}\frac{\partial h_2}{\partial\beta}\left(2\kappa_2-\frac{m-n}{m}\lambda_1\right)\right.$$
$$\left.-h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\left(2\lambda_1-\frac{m-n}{m}\kappa_2\right)+\frac{m+n}{m}\left\{-\frac{\partial\kappa_1}{\delta\alpha}+\frac{2}{h_2}\frac{\partial h_2}{\partial\alpha}\kappa_1\right\}\right]$$
$$+\int ds\,\delta w\,\lambda h_2\left[\frac{\partial}{\partial\alpha}\left\{\frac{h_1}{h_2}\left(2\lambda_1-\frac{m-n}{m}\kappa_2\right)\right\}-\frac{1}{h_2}\frac{\partial h_1}{\partial\alpha}\left(2\lambda_1-\frac{m-n}{m}\kappa_2\right)\right.$$
$$\left.+h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\left(2\kappa_2-\frac{m-n}{m}\lambda_1\right)+\frac{m+n}{m}\left\{-\frac{\partial\kappa_1}{\partial\beta}+\frac{2}{h_1}\frac{\partial h_1}{\partial\beta}\kappa_1\right\}\right]$$
$$+\int ds\,\delta u\,\frac{1}{\rho_1}\left[\lambda\left(2\lambda_1-\frac{m-n}{m}\kappa_2\right)-2\,\frac{m+n}{m}\mu\kappa_1\right]$$
$$+\int ds\,\delta v\,\frac{1}{\rho_2}\left[-\mu\left(2\kappa_2-\frac{m-n}{m}\lambda_1\right)-2\,\frac{m+n}{m}\lambda\kappa_1\right], \quad\quad (27)$$

where, by integration by parts, the second term becomes

$$-\int ds\,\delta w\,\frac{m+n}{m}\frac{\partial}{\partial s}\left\{\lambda\mu\,(\kappa_2+\lambda_1)+(\lambda^2-\mu^2)\,\kappa_1\right\}.$$

Again,

$$\iint\delta W_2\,dS=\iint 2\left(\frac{2m}{m+n}\sigma_1+\frac{m-n}{m+n}\sigma_2\right)\delta\sigma_1\frac{d\alpha\,d\beta}{h_1\,h_2}$$
$$+\iint 2\left(\frac{2m}{m+n}\sigma_2+\frac{m-n}{m+n}\sigma_1\right)\delta\sigma_2\frac{d\alpha\,d\beta}{h_1\,h_2}+\iint\varpi\,\delta\varpi\,\frac{d\alpha\,d\beta}{h_1\,h_2}$$
$$=\iint d\alpha\,d\beta\,\delta u\left[-2\frac{\partial}{\partial\alpha}\left\{\left(\frac{1}{h_2}\right)\left(\frac{2m}{m+n}\sigma_1+\frac{m-n}{m+n}\sigma_2\right)\right\}\right.$$
$$\left.+2\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\left(\frac{2m}{m+n}\sigma_2+\frac{m-n}{m+n}\sigma_1\right)-h_1\frac{\partial}{\partial\beta}\left(\frac{\varpi}{h_1^2}\right)\right]$$
$$+\iint d\alpha\,d\beta\,\delta v\left[-2\frac{\partial}{\partial\beta}\left\{\left(\frac{1}{h_1}\right)\left(\frac{2m}{m+n}\sigma_2+\frac{m-n}{m+n}\sigma_1\right)\right\}\right.$$
$$\left.+2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\left(\frac{2m}{m+n}\sigma_1+\frac{m-n}{m+n}\sigma_2\right)-h_2\frac{\partial}{\partial\alpha}\left(\frac{\varpi}{h_2^2}\right)\right]$$
$$+\iint d\alpha\,d\beta\,\delta w\left[\left(\frac{2m}{m+n}\sigma_1+\frac{m-n}{m+n}\sigma_2\right)\frac{2}{h_1\,h_2\,\rho_1}+\left(\frac{2m}{m+n}\sigma_2+\frac{m-n}{m+n}\sigma_1\right)\frac{2}{h_1\,h_2\,\rho_2}\right]$$
$$+\int ds\,\lambda h_2\,\delta u\,\frac{2}{h_2}\left(\frac{2m}{m+n}\sigma_1+\frac{m-n}{m+n}\sigma_2\right)+\mu\,h_1\,\delta v\,\frac{2}{h_1}\left(\frac{2m}{m+n}\sigma_2+\frac{m-n}{m+n}\sigma_1\right)$$
$$+\int ds\left(\lambda h_2\,\delta v\,\frac{\varpi}{h_2}+\mu h_1\,\delta u\,\frac{\varpi}{h_1}\right), \quad\quad (28)$$

where the line-integral part is

$$\int ds \left[2\lambda\left(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\right) + \mu\varpi\right]\delta u + \int ds\left[2\mu\left(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\right) + \lambda\varpi\right]\delta v.$$

Again,

$$\iint\left(\frac{\partial^2 l_3}{\partial t^2}\delta l_3 + \frac{\partial^2 m_3}{\partial t^2}\delta m_3 + \frac{\partial^2 n_3}{\partial t^2}\delta n_3\right)dS$$

$$= \iint\left(-\frac{1}{h_2\rho_1}\frac{\partial^3 w}{\partial\alpha\,\partial t^2} + \frac{1}{h_1 h_2\rho_1^2}\frac{\partial^2 u}{\partial t^2}\right)\delta u\,d\alpha\,d\beta + \iint\left(-\frac{1}{h_1\rho_2}\frac{\partial^3 w}{\partial\beta\,\partial t^2} + \frac{1}{h_1 h_2\rho_2^2}\frac{\partial^2 v}{\partial t^2}\right)\delta v\,d\alpha\,d\beta$$

$$+ \iint\left[\frac{\partial}{\partial\alpha}\left(-\frac{h_1}{h_2}\frac{\partial^3 w}{\partial\alpha\,\partial t^2} + \frac{1}{h_2\rho_1}\frac{\partial^2 u}{\partial t^2}\right) + \frac{\partial}{\partial\beta}\left(-\frac{h_2}{h_1}\frac{\partial^3 w}{\partial\beta\,\partial t^2} + \frac{1}{\rho_2}\frac{\partial^2 v}{\partial t^2}\right)\right]\delta w\,d\alpha\,d\beta$$

$$- \int\left[\lambda\left(-h_1\frac{\partial^3 w}{\partial\alpha\,\partial t^2} + \frac{1}{\rho_1}\frac{\partial^2 u}{\partial t^2}\right) + \mu\left(-h_2\frac{\partial^3 w}{\partial\beta\,\partial t^2} + \frac{1}{\rho_2}\frac{\partial^2 v}{\partial t^2}\right)\right]\delta w\,ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

10. Collecting the terms, we have the differential equations of motion

$$\left[-\frac{X}{h_1h_2} - \frac{M}{\rho_1 h_1 h_2} + \frac{2\rho h}{h_1h_2}\frac{\partial^2 u}{\partial t^2}\right] + \tfrac{2}{3}\rho h^3\left(\frac{1}{h_1h_2\rho_1^2}\frac{\partial^2 u}{\partial t^2} - \frac{1}{h_2\rho_1}\frac{\partial^3 w}{\partial\alpha\,\partial t^2}\right)$$

$$- \tfrac{4}{3}nh^3\frac{m}{m+n}\left[\frac{1}{\rho_1}\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\left(2\kappa_2 - \frac{m-n}{m}\lambda_1\right)\right.$$

$$\left. + \frac{1}{\rho_1}\frac{\partial}{\partial\alpha}\left\{\frac{1}{h_2}\left(2\lambda_1 - \frac{m-n}{m}\kappa_2\right)\right\} - 2\frac{m+n}{m}\left\{\frac{\partial}{\partial\beta}\left(\frac{\kappa_1}{h_1\rho_1}\right) + \frac{\kappa_1}{\rho_1}\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\right\}\right]$$

$$+ 2nh\left[-2\frac{\partial}{\partial\alpha}\left\{\left(\frac{1}{h_2}\right)\left(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\right)\right\}\right.$$

$$\left. + 2\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\left(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\right) - h_1\frac{\partial}{\partial\beta}\left(\frac{\varpi}{h_1^2}\right)\right] = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$\left[-\frac{Y}{h_1h_2} + \frac{L}{\rho_2 h_1 h_2} + \frac{2\rho h}{h_1h_2}\frac{\partial^2 v}{\partial t^2}\right] + \tfrac{2}{3}\rho h^3\left(\frac{1}{h_1h_2\rho_2^2}\frac{\partial^2 v}{\partial t^2} - \frac{1}{h_1\rho_2}\frac{\partial^3 w}{\partial\beta\,\partial t^2}\right)$$

$$+ \tfrac{4}{3}nh^3\frac{m}{m+n}\left[\frac{1}{\rho_2}\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\left(2\lambda_1 - \frac{m-n}{m}\kappa_2\right)\right.$$

$$\left. + \frac{1}{\rho_2}\frac{\partial}{\partial\beta}\left\{\frac{1}{h_1}\left(2\kappa_2 - \frac{m-n}{m}\lambda_1\right)\right\} + 2\frac{m+n}{m}\left\{\frac{\partial}{\partial\alpha}\left(\frac{\kappa_1}{h_2\rho_2}\right) + \frac{\kappa_1}{\rho_2}\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\right\}\right]$$

$$+ 2nh\left[-2\frac{\partial}{\partial\beta}\left\{\frac{1}{h_1}\left(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\right)\right\}\right.$$

$$\left. + 2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\left(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\right) - h_2\frac{\partial}{\partial\alpha}\left(\frac{\varpi}{h_2^2}\right)\right] = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$\left[-\frac{Z}{h_1h_2}-\frac{\partial}{\partial\alpha}\left(\frac{M}{h_2}\right)+\frac{\partial}{\partial\beta}\left(\frac{L}{h_1}\right)+\frac{2\rho h}{h_1h_2}\frac{\partial^2 w}{\partial t^2}\right]$$

$$+\tfrac{2}{3}\rho h^3\left\{\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2\rho_1}\frac{\partial^2 u}{\partial t^2}-\frac{h_1}{h_2}\frac{\partial^3 w}{\partial\alpha\,\partial t^2}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{h_1\rho_2}\frac{\partial^2 v}{\partial t^2}-\frac{h_2}{h_1}\frac{\partial^3 w}{\partial\beta\,\partial t^2}\right)\right\}$$

$$+\tfrac{4}{3}nh^3\frac{m}{m+n}\left[\frac{\partial^2}{\partial\beta^2}\left\{\frac{h_2}{h_1}\left(2\kappa_2-\frac{m-n}{m}\lambda_1\right)\right\}-\frac{\partial^2}{\partial\alpha^2}\left\{\frac{h_1}{h_2}\left(2\lambda_1-\frac{m-n}{m}\kappa_2\right)\right\}\right.$$

$$-\frac{\partial}{\partial\beta}\left\{\frac{1}{h_1}\frac{\partial h_2}{\partial\beta}\left(2\kappa_2-\frac{m-n}{m}\lambda_1\right)\right\}+\frac{\partial}{\partial\alpha}\left\{\frac{1}{h_2}\frac{\partial h_1}{\partial\alpha}\left(2\lambda_1-\frac{m-n}{m}\kappa_2\right)\right\}$$

$$-\frac{\partial}{\partial\alpha}\left\{h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\left(2\kappa_2-\frac{m-n}{m}\lambda_1\right)\right\}+\frac{\partial}{\partial\beta}\left\{h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\left(2\lambda_1-\frac{m-n}{m}\kappa_2\right)\right\}$$

$$\left.+2\frac{m+n}{m}\left\{\frac{\partial^2\kappa_1}{\partial\alpha\,\partial\beta}-\frac{\partial}{\partial\beta}\left(\frac{\kappa_1}{h_2}\frac{\partial h_2}{\partial\alpha}\right)-\frac{\partial}{\partial\alpha}\left(\frac{\kappa_1}{h_1}\frac{\partial h_1}{\partial\beta}\right)\right\}\right]$$

$$+2nh\frac{2}{h_1h_2}\left\{\frac{1}{\rho_1}\left(\frac{2m}{m+n}\sigma_1+\frac{m-n}{m+n}\sigma_2\right)+\frac{1}{\rho_2}\left(\frac{2m}{m+n}\sigma_2+\frac{m-n}{m+n}\sigma_1\right)\right\}=0. \quad . \quad . \quad (32)$$

The first terms in these equations reduce to those in CLEBSCH's equations ('Elasticität,' pp. 306–307) in case the shell becomes a plane plate.

The second terms (in ρh^3) arise from the "rotatory inertia."

The third terms (in h^3n) arise from the term W_1 in the potential energy, and depend on the bending; the fourth terms (in $2nh$) arise from the term W_2, and depend on the stretching.

11. The boundary-conditions are

$$\left.\begin{aligned}-\left(A+\frac{V}{\rho_1}\right)+\tfrac{4}{3}nh^3\frac{m}{m+n}\frac{1}{\rho_1}\left[\lambda\left(2\lambda_1-\frac{m-n}{m}\kappa_2\right)-2\frac{m+n}{m}\mu\kappa_1\right]&\\+2nh\left[2\lambda\left(\frac{2m}{m+n}\sigma_1+\frac{m-n}{m+n}\sigma_2\right)+\mu\varpi\right]&=0,\\-B+\frac{U}{\rho_2}+\tfrac{4}{3}nh^3\frac{m}{m+n}\frac{1}{\rho_2}\left[-\mu\left(2\kappa_2-\frac{m-n}{m}\lambda_1\right)-2\frac{m+n}{m}\lambda\kappa_1\right]&\\+2nh\left[2\mu\left(\frac{2m}{m+n}\sigma_2+\frac{m-n}{m+n}\sigma_1\right)+\lambda\varpi\right]&=0.\end{aligned}\right\}\quad(33)$$

$$(\mu L - \lambda M) - C + \frac{\partial}{\partial s}(\lambda U + \mu V) + \tfrac{2}{3}\rho h^3\left[\lambda\left(h_1\frac{\partial^3 w}{\partial\alpha\,\partial t^2} - \frac{1}{\rho_1}\frac{\partial^2 u}{\partial t^2}\right) + \mu\left(h_2\frac{\partial^3 w}{\partial\beta\,\partial t^2} - \frac{1}{\rho_2}\frac{\partial^2 v}{\partial t^2}\right)\right]$$

$$+ \tfrac{4}{3}nh^3\frac{m}{m+n}\left\{\mu h_1\left[-\frac{\partial}{\partial\beta}\left\{\frac{h_2}{h_1}\left(2\kappa_2 - \frac{m-n}{m}\lambda_1\right)\right\} + \frac{1}{h_1}\frac{\partial h_2}{\partial\beta}\left(2\kappa_2 - \frac{m-n}{m}\lambda_1\right) - h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\left(2\lambda_1 - \frac{m-n}{m}\kappa_2\right)\right]\right.$$

$$+ \lambda h_2\left[\frac{\partial}{\partial\alpha}\left\{\frac{h_1}{h_2}\left(2\lambda_1 - \frac{m-n}{m}\kappa_2\right)\right\} - \frac{1}{h_2}\frac{\partial h_1}{\partial\alpha}\left(2\lambda_1 - \frac{m-n}{m}\kappa_2\right) + h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\left(2\kappa_2 - \frac{m-n}{m}\lambda_1\right)\right]$$

$$- \frac{m+n}{m}\frac{\partial}{\partial s}[\lambda\mu(\kappa_2 + \lambda_1) + (\lambda^2 - \mu^2)\kappa_1]$$

$$\left. + \frac{m+n}{m}\left[h_1\mu\left(-\frac{\partial\kappa_1}{\partial\alpha} + \frac{2}{h_2}\frac{\partial h_2}{\partial\alpha}\kappa_1\right) + h_2\lambda\left(-\frac{\partial\kappa_1}{\partial\beta} + \frac{2}{h_1}\frac{\partial h_1}{\partial\beta}\kappa_1\right)\right]\right\} = 0, \quad (34)$$

$$\lambda V - \mu U + \tfrac{4}{3}nh^3\frac{m}{m+n}\left[\mu^2\left(2\kappa_2 - \frac{m-n}{m}\lambda_1\right) - \lambda^2\left(2\lambda_1 - \frac{m-n}{m}\kappa_2\right) + 2\lambda\mu\frac{m+n}{m}\kappa_1\right] = 0. \quad (35)$$

The first terms in each of these equations are the same as those in CLEBSCH's equations, pp. 306, 307.

The couple $-[\lambda U + \mu V]$ is that called by DE ST. VENANT the moment of torsion; the couple $\lambda V - \mu U$ is that called by him the moment of flexure, and their axes are the normal and tangent to the edge respectively. The former of these may be considered as arising from a distribution of force in lines normal to the middle-surface and in the edge; the difference of the forces in consecutive elements gives rise to a resultant force normal to the middle-surface which coalesces with C. This is the explanation of the union of two of the boundary-conditions given by POISSON in one.

We are going to apply the equations just developed to determine the small free vibrations of the shell. The terms depending on the rotatory inertia will be neglected.

§ 6. *Possibility of Certain Modes of Vibration.*

12. Now let us suppose, if possible, that the shell vibrates in such a manner that no line on the middle-surface is altered in length. This requires that σ_1, σ_2, ϖ be all zero. Thus, from equations (13) we derive

$$h_1\frac{\partial u}{\partial\alpha} + h_1 h_2 v\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + \frac{w}{\rho_1} = 0,$$

$$h_2\frac{\partial v}{\partial\beta} + h_1 h_2 u\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right) + \frac{w}{\rho_2} = 0,$$

$$\frac{h_2}{h_1}\frac{\partial}{\partial\beta}(h_1 u) + \frac{h_1}{h_2}\frac{\partial}{\partial\alpha}(h_2 v) = 0.$$

These are three partial differential equations to determine the forms of u, v, w; and, if either u and w or v and w be eliminated, they will in general lead to an equation of the third order to determine v or u. When one of these is determined, the rest are known. But at the edge we have to satisfy four boundary-conditions, and this will not be generally possible with solutions of a system of equations such as the above.

13. Since σ_1, σ_2, ϖ may not in general be regarded as of a higher order of small quantities than κ_2, λ_1, κ_1, it follows that the term in W_2 in the potential energy which contains h as a factor is very great compared with the term in W_1 which contains h^3, and we may form approximate equations of vibration and boundary-conditions by omitting the latter term.

The equations of motion thus formed are—

$$\left.\begin{aligned}
0 = \frac{\partial^2 u}{\partial t^2} + h_1 h_2 \frac{n}{\rho}\Big[-2\frac{\partial}{\partial\alpha}\Big\{\frac{1}{h_2}\Big(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\Big)\Big\} \\
+ 2\frac{\partial}{\partial\alpha}\Big(\frac{1}{h_2}\Big)\Big(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\Big) - h_1\frac{\partial}{\partial\beta}\Big(\frac{\varpi}{h_1^2}\Big)\Big], \\
0 = \frac{\partial^2 v}{\partial t^2} + h_1 h_2 \frac{n}{\rho}\Big[-2\frac{\partial}{\partial\beta}\Big\{\frac{1}{h_1}\Big(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\Big)\Big\} \\
+ 2\frac{\partial}{\partial\beta}\Big(\frac{1}{h_1}\Big)\Big(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\Big) - h_2\frac{\partial}{\partial\alpha}\Big(\frac{\varpi}{h_2^2}\Big)\Big], \\
0 = \frac{\partial^2 w}{\partial t^2} + 2\frac{n}{\rho}\Big[\frac{1}{\rho_1}\Big(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\Big) + \frac{1}{\rho_2}\Big(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\Big)\Big].
\end{aligned}\right\} \quad (36)$$

And the boundary-conditions are—

$$\left.\begin{aligned}
2\lambda\Big(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\Big) + \mu\varpi = 0, \\
2\mu\Big(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\Big) + \lambda\varpi = 0.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

(1.) Let us examine the possibility* of purely normal vibrations.

Since $u = 0$, $v = 0$, the equations of motion become simply

$$\frac{\partial^2 w}{\partial t^2} + 2\frac{n}{\rho}\frac{2m}{m+n}\Big(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \frac{2\sigma}{\rho_1\rho_2}\Big)w = 0, \quad . \quad . \quad . \quad . \quad . \quad (38)$$

where $\sigma = (m-n)/2m$ is the ratio of linear lateral contraction to linear longitudinal extension of the material of the shell.

* MATHIEU convinces himself of the impossibility by general reasoning.

In order that all parts of the system may be in the same phase, it is necessary that $1/\rho_1^2 + 1/\rho_2^2 + 2\sigma/\rho_1\rho_2 =$ const. all over the surface.

Again, in the u, v equations we must pick out the terms containing w, and, observing that w is independent of α, β, we may write them—

$$\left.\begin{aligned} -\frac{\partial}{\partial\alpha}\left\{\frac{1}{h_2}\left(\frac{1}{\rho_1}+\frac{\sigma}{\rho_2}\right)\right\} + \frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\left(\frac{1}{\rho_2}+\frac{\sigma}{\rho_1}\right) = 0, \\ -\frac{\partial}{\partial\beta}\left\{\frac{1}{h_1}\left(\frac{1}{\rho_2}+\frac{\sigma}{\rho_1}\right)\right\} + \frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\left(\frac{1}{\rho_1}+\frac{\sigma}{\rho_2}\right) = 0. \end{aligned}\right\}$$

Thus,

$$(1-\sigma)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right) = -\frac{1}{h_2}\frac{\partial}{\partial\alpha}\left(\frac{1}{\rho_1}+\frac{\sigma}{\rho_2}\right),$$

and,

$$(1-\sigma)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) = \frac{1}{h_1}\frac{\partial}{\partial\beta}\left(\frac{1}{\rho_2}+\frac{\sigma}{\rho_1}\right).$$

But, by equations (17),

$$\left.\begin{aligned} \left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right) - \frac{1}{h_2}\frac{\partial}{\partial\alpha}\left(\frac{1}{\rho_2}\right) = 0, \\ \left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + \frac{1}{h_1}\frac{\partial}{\partial\beta}\left(\frac{1}{\rho_1}\right) = 0. \end{aligned}\right\}$$

Substituting, we get

$$\left.\begin{aligned} \frac{1}{h_2}\frac{\partial}{\partial\alpha}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) = 0, \\ \frac{1}{h_1}\frac{\partial}{\partial\beta}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) = 0. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (39)$$

So that $1/\rho_1 + 1/\rho_2 =$ const. all over the surface.

The two conditions of possibility of normal vibrations show that the middle-surface must have both its principal radii of curvature constant at every point. These conditions are satisfied by the sphere, the circular cylinder, and the plane.

Again, if the surface be bounded by an edge, we have, since $\varpi = 0$, $\lambda(1/\rho_1 + \sigma/\rho_2) = 0$, $\mu(1/\rho_2 + \sigma/\rho_1) = 0$; these can coexist for all values of λ, μ if $\sigma^2 - 1 = 0$, and $1/\rho_1 = \pm 1/\rho_2$.

To make n positive, or the material resist distortion, we must have $\frac{1}{2} - \sigma$ positive, so that σ cannot be $= 1$; the equation $\sigma = -1$ makes $n = 3m = 3k + n$, so that $k = 0$ or the material of the shell would offer no resistance to compression; thus, the equations above written cannot coexist for all values of λ, μ, and hence one of the two λ, μ must be zero, and one of the two equations $1/\rho_2 + \sigma/\rho_1 = 0$ and $1/\rho_1 + \sigma/\rho_2 = 0$, must hold at the edge. These conditions cannot be satisfied on a sphere or cylinder.

The complete spherical shell may execute purely radial vibrations, and the frequency is

$$\frac{1}{\pi}\sqrt{\frac{(1+\sigma)\,n}{(1-\sigma)\,a^2\rho}},$$

where a is the radius.*

The indefinitely long circular cylinder may also execute purely radial vibrations with a frequency

$$\frac{1}{2\pi}\sqrt{\frac{2n}{(1-\sigma)\,a^2\rho}},$$

a being the radius.

Observing that the more accurate equations of motion and boundary-conditions, which contain the terms in h^3, will in all such terms have only differential coefficients of w with respect to α or β, the above theory is seen to hold also if these more accurate equations be considered.

(2.) Again, consider the possibility † of purely tangential vibrations, the edge being a line of curvature.

Since $w = 0$, the third of equations (36) gives

$$(\sigma_1 + \sigma\sigma_2)/\rho_1 + (\sigma_2 + \sigma\sigma_1)/\rho_2 = 0$$

at all points of the surface.

Now, the boundary-conditions at $\alpha =$ const. are

$$\left.\begin{aligned} \sigma_1 + \sigma\sigma_2 &= 0, \\ \varpi &= 0, \end{aligned}\right\}$$

and with two functions u, v it will not generally be possible to satisfy these conditions.

If, however, the surface be of revolution, and β be the longitude, then $\partial h_1^{-1}/\partial\beta = 0$, and all the conditions can be satisfied by taking

$$\left.\begin{aligned} &(1)\ u = 0, \\ &(2)\ \frac{\partial v}{\partial\beta} = 0 \end{aligned}\right\}\ \text{at all points of the surface,}$$

$$(3)\ \frac{\partial}{\partial\alpha}(h_2 v) = 0 \text{ at the edge;}$$

* [In the paper as read, this result was verified by reference to a question set in the Mathematical Tripos, part III., 1885. It has since been pointed out to me that it coincides with the formula given by LAMB in 'London Math. Soc. Proc.,' vol. 14, p. 50.—July, 1888.]

† MATHIEU deduced the possibility of some purely tangential vibrations from his differential equations.

and the equation of motion is

$$\frac{\rho}{n}\frac{\partial^2 v}{\partial t^2} + h_1 h_2^2 \frac{\partial}{\partial \alpha}\left\{\frac{h_1}{h_2^3}\frac{\partial}{\partial \alpha}(h_2 v)\right\} = 0. \quad \ldots \ldots \quad (40)$$

Hence a general theorem :—For any surface of revolution there exists a system of symmetrical vibrations, in which every element moves perpendicular to the meridian plane through a distance which is the same for all points on a parallel of latitude, and the frequency of such vibrations depends only on the rigidity of the material, while the ratios of the intervals are independent of the material. These are the only purely tangential vibrations of which the shell is capable.

14. Let us examine more minutely the question whether a spherical shell can vibrate in such a manner that no line on the middle-surface is altered in length.

Taking $\alpha = \theta$, $\beta = \phi$, the colatitude and longitude of a point on the middle-surface, and a the radius, $h_1 = 1/a$, $h_2 = 1/(a \sin \theta)$, thus

$$\left.\begin{aligned} a\sigma_1 &= \frac{\partial u}{\partial \theta} + w, \\ a\sigma_2 &= w + u \cot \theta + \frac{1}{\sin \theta}\frac{\partial v}{\partial \phi}, \\ a\varpi &= \frac{\partial}{\partial \phi}\left(\frac{u}{\sin \theta}\right) + \sin \theta \frac{\partial}{\partial \theta}\left(\frac{v}{\sin \theta}\right); \end{aligned}\right\} \quad \ldots \ldots \quad (41)$$

$$\left.\begin{aligned} a^2\kappa_2 &= \frac{1}{\sin^2 \theta}\frac{\partial^2 w}{\partial \phi^2} + \cot \theta \frac{\partial w}{\partial \theta} - \frac{1}{\sin \theta}\frac{\partial v}{\partial \phi} - u \cot \theta, \\ -a^2\lambda_1 &= \frac{\partial^2 w}{\partial \theta^2} - \frac{\partial u}{\partial \theta}, \\ a^2\kappa_1 &= \frac{1}{\sin \theta}\frac{\partial^2 w}{\partial \theta\, \partial \phi} - \frac{\cos \theta}{\sin^2 \theta}\frac{\partial w}{\partial \phi} - \frac{\partial v}{\partial \theta} - \frac{1}{\sin \theta}\frac{\partial u}{\partial \phi} + v \cot \theta. \end{aligned}\right\} \quad \ldots \quad (42)$$

Suppose σ_1, σ_2, ϖ all zero, then

$$w = -\frac{\partial u}{\partial \theta},$$

and

$$\sin \theta \frac{\partial}{\partial \theta}\left(\frac{u}{\sin \theta}\right) = \frac{\partial}{\partial \phi}\left(\frac{v}{\sin \theta}\right),$$

$$\sin \theta \frac{\partial}{\partial \theta}\left(\frac{v}{\sin \theta}\right) = -\frac{\partial}{\partial \phi}\left(\frac{u}{\sin \theta}\right).$$

These are the conditions given by Lord RAYLEIGH, and they show that $u \operatorname{cosec} \theta$ and $v \operatorname{cosec} \theta$ are conjugate solutions of the equation

$$\frac{\partial^2 X}{\partial\phi^2} + \left(\sin\theta\frac{\partial}{\partial\theta}\right)^2 X = 0\,; \qquad (43)$$

and w is given by the equation

$$w = -\frac{\partial u}{\partial\theta}.$$

Substituting from σ_1, σ_2, $\varpi = 0$, we find

$$a^2\kappa_2 = \frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\phi^2} + \cot\theta\frac{\partial w}{\partial\theta} + w,$$

$$a^2\lambda_1 = \frac{\partial^3 u}{\partial\theta^3} + \frac{\partial u}{\partial\theta},$$

$$a^2\kappa_1 = -\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial^2 u}{\partial\theta\,\partial\phi}\right).$$

Now,

$$\frac{\partial^2 u}{\partial\phi^2} = -\sin\theta\left(\sin\theta\frac{\partial}{\partial\theta}\right)^2\left(\frac{u}{\sin\theta}\right).$$

Thus,

$$\frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\phi^2} = -\frac{1}{\sin^2\theta}\frac{\partial^3 u}{\partial\theta\,\partial\phi^2} = \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\sin\theta\frac{\partial}{\partial\theta}\right)^2\left(\frac{u}{\sin\theta}\right)\right]$$

$$= \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left[\sin^2\theta\frac{\partial^2 u}{\partial\theta^2} - \sin\theta\cos\theta\frac{\partial u}{\partial\theta} + u\right]$$

$$= \frac{\partial^3 u}{\partial\theta^3} + \cot\theta\frac{\partial^2 u}{\partial\theta^2} + 2\frac{\partial u}{\partial\theta}.$$

Hence, $\kappa_2 = \lambda_1$.*

The boundary-conditions arising from the terms in δu, δv in Art. 11 are now

$$2\lambda(1-\sigma)\kappa_2 - 2\mu(1-\sigma)\kappa_1 = 0,$$

$$-2\mu(1-\sigma)\kappa_2 - 2\lambda(1-\sigma)\kappa_1 = 0\,;$$

since $\lambda^2 + \mu^2 = 1$, and $\frac{1}{2} - \sigma$ is positive, the only way of satisfying these equations is to take $\kappa_1 = 0$, $\kappa_2 = 0$ at the edge.

Now,

$$a^2\lambda_1 = a^2\kappa_2 = \frac{\partial^3 u}{\partial\theta^3} + \frac{\partial u}{\partial\theta}, \qquad a^2\kappa_1 = -\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial^2 u}{\partial\theta\,\partial\phi}\right).$$

And, as shown by Lord Rayleigh,

$$\left.\begin{aligned} u &= \sum_{s=2}^{s=\infty}\sin\theta\left[A_s\tan^s\frac{\theta}{2}\right]_{\cos s\phi}^{-\sin s\phi} \\ v &= \sum_{s=2}^{s=\infty}\sin\theta\left[A_s\tan^s\frac{\theta}{2}\right]_{\sin s\phi}^{\cos s\phi} \\ w &= \sum_{s=2}^{s=\infty}(s+\cos\theta)\left[A_s\tan^s\frac{\theta}{2}\right]_{-\cos s\phi}^{\sin s\phi} \end{aligned}\right\} \qquad (44)$$

* This might have been written down at once by the aid of Gauss's deformation theorem.

From (43)

$$\sin\theta\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\frac{u}{\sin\theta}\right)\right]+\frac{1}{\sin\theta}\frac{\partial^2 u}{\partial\phi^2}=0\,;$$

put

$$\sin\theta\tan^s\frac{\theta}{2}=u_0,$$

then

$$\frac{d^2u_0}{d\theta^2}+u_0=\cot\theta\frac{du_0}{d\theta}+u_0\left(s^2\operatorname{cosec}^2\theta-\cot^2\theta\right),$$

and

$$\frac{du_0}{d\theta}=\cot\theta\, u_0+\frac{s}{2}u_0\frac{\sec^2\frac{\theta}{2}}{\tan\frac{\theta}{2}}=u_0\left(\cot\theta+\frac{s}{\sin\theta}\right);$$

so that

$$\frac{d^2u_0}{d\theta^2}+u_0=u_0 s\frac{s+\cos\theta}{\sin^2\theta},$$

$$\frac{d^3u_0}{d\theta^3}+\frac{du_0}{d\theta}=u_0 s\left[-\frac{2s\cos\theta}{\sin^3\theta}-\frac{1}{\sin\theta}-\frac{2\cos^2\theta}{\sin^3\theta}\right]+\frac{s^2+s\cos\theta}{\sin^2\theta}u_0\left(\cot\theta+\frac{s}{\sin\theta}\right)$$

$$=u_0 s\frac{s^2-1}{\sin^3\theta}\,;$$

hence,

$$a^2\kappa_2=\sum_{s=2}^{s=\infty}\left[(s^3-s)\,\mathrm{A}_s\tan^s\frac{\theta}{2}\operatorname{cosec}^2\theta\right]_{\cos s\phi}^{-\sin s\phi}.$$

Again,

$$\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{du_0}{d\theta}\right)=\frac{d}{d\theta}\left[\frac{\cos\theta+s}{\sin^2\theta}u_0\right]$$

$$=\frac{(\cos\theta+s)^2}{\sin^3\theta}u_0-\left(\frac{2s\cos\theta}{\sin^3\theta}+\frac{1}{\sin\theta}+\frac{2\cos^2\theta}{\sin^3\theta}\right)u_0=u_0\frac{s^2-1}{\sin^3\theta}\,;$$

hence,

$$a^2\kappa_1=\sum_{s=2}^{s=\infty}\left[(s^3-s)\,\mathrm{A}_s\tan^s\frac{\theta}{2}\operatorname{cosec}^2\theta\right]_{\sin s\phi}^{\cos s\phi},$$

so that κ_2 and κ_1 cannot both vanish all along any curve drawn on the middle-surface, unless the A vanish, which gives no displacement.

We have shown explicitly in this particular case that the assumption that no line on the middle-surface is altered in length does lead to expressions for the displacements which cannot satisfy the boundary-conditions which hold at a free edge.

§7. *Vibrations of Spherical Shell.*

15. Let us now apply the equations of Art. (13) to the discussion of the vibrations of a spherical shell; we have

$$\left.\begin{aligned}
\frac{a^2\rho}{n}\sin\theta\frac{\partial^2 u}{\partial t^2} &= 2\frac{\partial}{\partial\theta}\left(\frac{2m}{m+n}\frac{\sigma_1}{h_2}+\frac{m-n}{m+n}\frac{\sigma_2}{h_2}\right)\\
&\quad - 2\frac{\partial}{\partial\theta}\left(\frac{1}{h_2}\right)\left(\frac{2m}{m+n}\sigma_2+\frac{m-n}{m+n}\sigma_1\right)+h_1\frac{\partial}{\partial\phi}\left(\frac{\varpi}{h_1^2}\right),\\
\frac{a^2\rho}{n}\sin\theta\frac{\partial^2 v}{\partial t^2} &= 2\frac{\partial}{\partial\phi}\left(\frac{2m}{m+n}\frac{\sigma_2}{h_1}+\frac{m-n}{m+n}\frac{\sigma_1}{h_1}\right)\\
&\quad - 2\frac{\partial}{\partial\phi}\left(\frac{1}{h_1}\right)\left(\frac{2m}{m+n}\sigma_1+\frac{m-n}{m+n}\sigma_2\right)+h_2\frac{\partial}{\partial\theta}\left(\frac{\varpi}{h_2^2}\right),\\
\frac{a^2\rho}{n}\sin\theta\frac{\partial^2 w}{\partial v^2} &= -2a\sin\theta\frac{3m-n}{m+n}(\sigma_1+\sigma_2)\cdot
\end{aligned}\right\} \quad (45)$$

In these we are to substitute for h_1, h_2 their values

$$h_1 = 1/a, \qquad h_2 = 1/(a\sin\theta)$$

and for σ_1, σ_2, ϖ their values from

$$a\sigma_1 = \frac{\partial u}{\partial\theta}+w$$

$$a\sigma_2 = w + u\cot\theta + \frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}$$

$$a\varpi = \frac{\partial}{\partial\phi}\left(\frac{u}{\sin\theta}\right)+\sin\theta\frac{\partial}{\partial\theta}\left(\frac{v}{\sin\theta}\right)\cdot$$

Let us take u, v, w as functions of t to be proportional to $e^{\iota pt}$, then the period is $2\pi/p$; also take $p^2a^2\rho = n\kappa^2$, where κ is a number, then we have the three equations—

$$\begin{aligned}
\kappa^2 u\sin\theta &+ 2\frac{2m}{m+n}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial u}{\partial\theta}+w\right)\right]+2\frac{m-n}{m+n}\frac{\partial}{\partial\theta}\left(w\sin\theta+u\cos\theta+\frac{\partial v}{\partial\phi}\right)\\
&- 2\frac{2m}{m+n}\cos\theta\left(w+u\cot\theta+\frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}\right)-2\frac{m-n}{m+n}\cos\theta\left(\frac{\partial u}{\partial\theta}+w\right)\\
&+\frac{1}{\sin\theta}\frac{\partial^2 u}{\partial\phi^2}+\frac{\partial^2 v}{\partial\theta\,\partial\phi}-\cot\theta\frac{\partial v}{\partial\phi}=0, \quad \ldots \quad (46)
\end{aligned}$$

$$\begin{aligned}
\kappa^2 v\sin\theta &+ 2\frac{2m}{m+n}\frac{\partial}{\partial\phi}\left(w+u\cot\theta+\frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}\right)+2\frac{m-n}{m+n}\frac{\partial}{\partial\phi}\left(w+\frac{\partial u}{\partial\theta}\right)\\
&+\frac{\partial^2 u}{\partial\theta\,\partial\phi}+\cot\theta\frac{\partial u}{\partial\phi}+\sin\theta\frac{\partial^2 v}{\partial\theta^2}+\cos\theta\frac{\partial v}{\partial\theta}-\frac{v}{\sin\theta}(\cos^2\theta-\sin^2\theta)=0, \quad (47)
\end{aligned}$$

$$\kappa^2 w = 2\frac{3m-n}{m+n}\left(\frac{\partial u}{\partial\theta}+u\cot\theta+\frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}+2w\right). \quad \ldots \quad (48)$$

The first two of these are—

$$\kappa^2 u + \left(1 + \frac{3m-n}{m+n}\right)\left(\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta}\right) + 2\frac{\partial w}{\partial\theta}\frac{3m-n}{m+n} - \left(1 + \frac{3m-n}{m+n}\right)u\cot^2\theta$$
$$+\left(1 - \frac{3m-n}{m+n}\right)u - \left(2 + \frac{3m-n}{m+n}\right)\frac{\cos\theta}{\sin^2\theta}\frac{\partial v}{\partial\phi} + \frac{3m-n}{m+n}\frac{1}{\sin\theta}\frac{\partial^2 v}{\partial\theta\,\partial\phi} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0, \quad (49)$$

$$\kappa^2 v + \left(\frac{\partial^2 v}{\partial\theta^2} + \cot\theta\frac{\partial v}{\partial\theta}\right) + v\,(2 - \operatorname{cosec}^2\theta) + \frac{2}{\sin\theta}\frac{3m-n}{m+n}\frac{\partial w}{\partial\phi} + \frac{\cos\theta}{\sin^2\theta}\frac{\partial u}{\partial\phi}\left(2 + \frac{3m-n}{m+n}\right)$$
$$+\frac{1}{\sin^2\theta}\left(1 + \frac{3m-n}{m+n}\right)\frac{\partial^2 v}{\partial\phi^2} + \frac{3m-n}{m+n}\frac{1}{\sin\theta}\frac{\partial^2 u}{\partial\theta\,\partial\phi} = 0. \quad . \quad . \quad . \quad . \quad (50)$$

Substituting from (48), these are—

$$\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta} + (2 + \kappa^2 - \operatorname{cosec}^2\theta)\,u + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v}{\partial\phi} + \frac{\kappa^2}{2}\frac{\partial w}{\partial\theta} = 0, \quad . \quad (51)$$

$$\frac{\partial^2 v}{\partial\theta^2} + \cot\theta\frac{\partial v}{\partial\theta} + (2 + \kappa^2 - \operatorname{cosec}^2\theta)\,v + \frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2} + \frac{2\cos\theta}{\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\kappa^2}{2\sin\theta}\frac{\partial w}{\partial\phi} = 0, \quad (52)$$

and, writing

$$\frac{c}{\kappa^2}\left(\kappa^2 - 4\frac{3m-n}{m+n}\right) = \frac{3m-n}{m+n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

(48) becomes

$$\frac{\kappa^2}{2}w = c\left(\frac{\partial u}{\partial\theta} + u\cot\theta + \frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Substituting for w, we find

$$(1+c)\left(\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$
$$+\left[\kappa^2 + 2 - (1+c)\operatorname{cosec}^2\theta\right]u - (2+c)\frac{\cos\theta}{\sin^2\theta}\frac{\partial v}{\partial\phi} + \frac{c}{\sin\theta}\frac{\partial^2 v}{\partial\theta\,\partial\phi} = 0, \quad (55)$$

$$\frac{\partial^2 v}{\partial\theta^2} + \cot\theta\frac{\partial v}{\partial\theta} + \frac{1+c}{\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}$$
$$+\left[\kappa^2 + 2 - \operatorname{cosec}^2\theta\right]v + (2+c)\frac{\cos\theta}{\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{c}{\sin\theta}\frac{\partial^2 u}{\partial\theta\,\partial\phi} = 0. \quad (56)$$

Since u, v, w must be the same for $\phi + 2\pi$ as for ϕ, we may put

$$u \propto \cos s\phi, \qquad v \propto \sin s\phi, \qquad w \propto \cos s\phi,$$

where s is an integer.

Then, for u, v, w as functions of θ it is convenient to take equations (51), (54), and (56), which become

$$\frac{d^2u}{d\theta^2} + \cot\theta\frac{du}{d\theta} + [2 + \kappa^2 - (1 + s^2)\operatorname{cosec}^2\theta]\,u - \frac{2s\cos\theta}{\sin^2\theta}v + \frac{\kappa^2}{2}\frac{dw}{d\theta} = 0, \quad . \quad (57)$$

$$\frac{d^2v}{d\theta^2} + \cot\theta\frac{dv}{d\theta} + [2 + \kappa^2 - \{1 + s^2(1 + c)\}\operatorname{cosec}^2\theta]\,v$$
$$- (2 + c)\frac{s\cos\theta}{\sin^2\theta}u - \frac{sc}{\sin\theta}\frac{du}{d\theta} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

$$\frac{\kappa^2}{2}w = c\left(\frac{du}{d\theta} + u\cot\theta + \frac{sv}{\sin\theta}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Differentiating (59) with respect to θ, dividing by c and subtracting from (57),

$$(2 + \kappa^2 - s^2\operatorname{cosec}^2\theta)\,u = \frac{sv\cos\theta}{\sin^2\theta} + \frac{s}{\sin\theta}\frac{dv}{d\theta} - \frac{\kappa^2}{2}\left(1 + \frac{1}{c}\right)\frac{dw}{d\theta}.$$

Write $u\sin\theta = \mathrm{U}$, $v\sin\theta = \mathrm{V}$, thus,

$$[(2 + \kappa^2)\sin^2\theta - s^2]\frac{\mathrm{U}}{\sin\theta} = s\frac{d\mathrm{V}}{d\theta} - \frac{\kappa^2}{2}\left(1 + \frac{1}{c}\right)\frac{dw}{d\theta}\sin^2\theta\,; \quad . \quad . \quad . \quad (60)$$

and (59) becomes

$$\frac{d\mathrm{U}}{d\theta} + \frac{s\mathrm{V}}{\sin\theta} = \frac{\kappa^2}{2c}w\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

We are going to substitute from (60) in (58) and (61); the result will enable us to eliminate V, and obtain an equation for w.

We have

$$\mathrm{U} = \frac{-\dfrac{\kappa^2}{2}\left(1 + \dfrac{1}{c}\right)\sin^3\theta\dfrac{dw}{d\theta} + s\,.\sin\theta\dfrac{d\mathrm{V}}{d\theta}}{(2 + \kappa^2)\sin^2\theta - s^2};$$

therefore

$$\frac{d\mathrm{U}}{d\theta} = -\frac{\kappa^2}{2}\left(1 + \frac{1}{c}\right)\frac{\sin^2\theta}{(2 + \kappa^2)\sin^2\theta - s^2}\left[\sin\theta\frac{d^2w}{d\theta^2} + 3\cos\theta\frac{dw}{d\theta} - \frac{2(2 + \kappa^2)\sin^2\theta\cos\theta}{(2 + \kappa^2)\sin^2\theta - s^2}\frac{d\theta}{dw}\right]$$
$$+ \frac{s\,.\sin\theta}{(2 + \kappa^2)\sin^2\theta - s^2}\left[\frac{d^2\mathrm{V}}{d\theta^2} + \cot\theta\frac{d\mathrm{V}}{d\theta} - \frac{2(2 + \kappa^2)\sin\theta\cos\theta}{(2 + \kappa^2)\sin^2\theta - s^2}\frac{d\mathrm{V}}{d\theta}\right].$$

Substituting in (61), we have, on multiplying by $\operatorname{cosec}\theta[(2 + \kappa^2)\sin^2\theta - s^2]^2$,

$$\frac{\kappa^2}{2}\left(1+\frac{1}{c}\right)\sin^2\theta\,[(2+\kappa^2)\sin^2\theta - s^2]\frac{d^2w}{d\theta^2}$$
$$+\cot\theta\,[(2+\kappa^2)\sin^2\theta - 3s^2]\frac{dw}{d\theta} + \frac{\kappa^2}{2c}w\,[(2+\kappa^2)\sin^2\theta - s^2]^2$$
$$- s\,[(2+\kappa^2)\sin^2\theta - s^2]\frac{d^2\mathrm{V}}{d\theta^2} + s\,[(2+\kappa^2)\sin^2\theta + s^2]\cot\theta\frac{d\mathrm{V}}{d\theta}$$
$$-\frac{s}{\sin^2\theta}[(2+\kappa^2)\sin^2\theta - s^2]^2\,\mathrm{V} = 0. \quad . \quad . \quad . \quad . \quad (62)$$

Now, substituting for $d\mathrm{U}/d\theta$ from (61) and for U from (60), (58) becomes

$$\frac{1}{\sin\theta}\left(\frac{d^2\mathrm{V}}{d\theta^2} - \cot\theta\frac{d\mathrm{V}}{d\theta} + \mathrm{V}\operatorname{cosec}^2\theta\right) + [2+\kappa^2 - \{1+s^2(1+c)\}\operatorname{cosec}^2\theta]\frac{\mathrm{V}}{\sin\theta}$$
$$-\frac{sc}{\sin^2\theta}\left(\frac{\kappa^2}{2c}w\sin\theta - \frac{s}{\sin\theta}\mathrm{V}\right) - \frac{2s\cos\theta}{\sin^2\theta}\,\frac{-\frac{\kappa^2}{2}\left(1+\frac{1}{c}\right)\sin^2\theta\frac{dw}{d\theta} + s\frac{d\mathrm{V}}{d\theta}}{(2+\kappa^2)\sin^2\theta - s^2} = 0,$$

or, multiplying by $\sin\theta\,[(2+\kappa^2)\sin^2\theta - s^2]$,

$$[(2+\kappa^2)\sin^2\theta - s^2]\frac{d^2\mathrm{V}}{d\theta^2} - [(2+\kappa^2)\sin^2\theta + s^2]\cot\theta\frac{d\mathrm{V}}{d\theta} + [(2+\kappa^2)\sin^2\theta - s^2]^2\frac{\mathrm{V}}{\sin^2\theta}$$
$$-\frac{\kappa^2}{2}s.w\,[(2+\kappa^2)\sin^2\theta - s^2] + \kappa^2 s\left(1+\frac{1}{c}\right)\sin\theta\cos\theta\frac{dw}{d\theta} = 0.$$

Multiply this by s and add it to (62), thus,

$$\frac{\kappa^2}{2}\left(1+\frac{1}{c}\right)\left[\sin^2\theta\frac{d^2w}{d\theta^2} + \sin\theta\cos\theta\frac{dw}{d\theta}\right] + \left\{[(2+\kappa^2)\sin^2\theta - s^2]\frac{\kappa^2}{2c} - \frac{\kappa^2}{2}s^2\right\}w = 0,$$

or

$$\frac{d^2w}{d\theta^2} + \cot\theta\frac{dw}{d\theta} + \left(\frac{2+\kappa^2}{1+c} - \frac{s^2}{\sin^2\theta}\right)w = 0. \quad . \quad . \quad . \quad . \quad (63)$$

Also, between (60) and (61), eliminate V, then

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\mathrm{U}}{d\theta}\right) + \frac{\mathrm{U}}{\sin\theta}[(2+\kappa^2)\sin^2\theta - s^2] = \frac{\kappa^2}{c}\sin\theta\cos\theta\,w - \frac{\kappa^2}{2}\sin\theta\frac{dw}{d\theta}. \quad (64)$$

The equations we have to satisfy are (61), (63), and (64). Writing μ instead of $\cos\theta$, these become

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dw}{d\mu}\right]+\left(\frac{2+\kappa^2}{1+c}-\frac{s^2}{1-\mu^2}\right)w=0, \quad . \quad . \quad . \quad . \quad . \quad (65)$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\mathrm{U}}{d\mu}\right]+\left(2+\kappa^2-\frac{s^2}{1-\mu^2}\right)\mathrm{U}=\kappa^2\left[\frac{\mu}{c}w+\tfrac{1}{2}(1-\mu^2)\frac{dw}{d\mu}\right], \qquad (66)$$

$$s\mathrm{V}=(1-\mu^2)\left(\frac{\kappa^2}{2c}w+\frac{d\mathrm{U}}{d\mu}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

Of these (67) gives V when U and w are known. The solution of (66) consists of two parts—one, the complementary function which satisfies (66) when $w=0$; the other, the particular integral which satisfies (66) when w is a solution of (65). We may show first that this particular integral is proportional to $(1-\mu^2)(dw/d\mu)$; take it to be $\lambda(1-\mu^2)(dw/d\mu)$.

For, writing (65) in the form

$$(1-\mu^2)^2\frac{d^2w}{d\mu^2}-2\mu(1-\mu^2)\frac{dw}{d\mu}-s^2w=-\frac{2+\kappa^2}{1+c}(1-\mu^2)w,$$

and differentiating, we have

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dw}{d\mu}\right\}\right]-\frac{s^2}{1-\mu^2}\left\{(1-\mu^2)\frac{dw}{d\mu}\right\}$$
$$=-\frac{2+\kappa^2}{1+c}\left[(1-\mu^2)\frac{dw}{d\mu}-2\mu w\right],$$

and the left-hand side is found by using (66) to be

$$-\left[\left(2+\kappa^2-\tfrac{1}{2}\frac{\kappa^2}{\lambda}\right)(1-\mu^2)\frac{dw}{d\mu}-\frac{\kappa^2}{\lambda c}\mu w\right],$$

so that $\lambda(1-\mu^2)(dw/d\mu)$ is a particular integral of (66), if

$$\kappa^2/2c\lambda=(2+\kappa^2)/(1+c)=2+\kappa^2-\kappa^2/2\lambda,$$

which are both satisfied by

$$\lambda=\frac{\kappa^2}{2c}\frac{1+c}{2+\kappa^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Thus,

$$\mathrm{U}=\frac{\kappa^2}{2+\kappa^2}\frac{1+c}{2c}(1-\mu^2)\frac{dw}{d\mu}$$

is a particular integral of (66).

16. We have now to consider the complementary functions.

In equations (65), (66), write

$$(2+\kappa^2)/(1+c)=\alpha(\alpha+1), \qquad 2+\kappa^2=\beta(\beta+1),$$

then these will be the equations of tesseral harmonics of orders α, β respectively.

Calling $T_\alpha^{(s)}(\mu)$ the solution which does not become infinite for $\mu=1$, we have

$$w=AT_\alpha^{(s)}(\mu),$$

$$U=BT_\beta^{(s)}(\mu)+\lambda A(1-\mu^2)\,.\,\frac{d}{d\mu}\{T_\alpha^{(s)}(\mu)\}\,.$$

To find V we have

$$\frac{dU}{d\mu}=B\frac{d}{d\mu}\{T_\beta^{(s)}(\mu)\}+\lambda\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dw}{d\mu}\right\}=B\frac{d}{d\mu}\{T_\beta^{(s)}(\mu)\}+\lambda\left(\frac{s^2}{1-\mu^2}-\frac{2+\kappa^2}{1+c}\right)w,$$

so that

$$V=\frac{B}{s}(1-\mu^2)\frac{d}{d\mu}\{T_\beta^{(s)}(\mu)\}+s\lambda AT_\alpha^{(s)}(\mu).$$

Hence,

$$\left.\begin{aligned}u&=\left[\lambda A\sqrt{(1-\mu^2)}\frac{d}{d\mu}\{T_\alpha^{(s)}(\mu)\}+\frac{B}{\sqrt{(1-\mu^2)}}T_\beta^{(s)}(\mu)\right]\cos s\phi\, e^{\iota pt},\\ v&=\left[\frac{s\lambda A}{\sqrt{(1-\mu^2)}}T_\alpha^{(s)}(\mu)+\frac{B}{s}\sqrt{(1-\mu^2)}\frac{d}{d\mu}\{T_\beta^{(s)}(\mu)\}\right]\sin s\phi\, e^{\iota pt},\\ w&=[AT_\alpha^{(s)}(\mu)]\cos s\phi\, e^{\iota pt}.\end{aligned}\right\} \quad . \quad (69)$$

17. *Properties of* $T_\alpha^{(s)}(\mu)$.

The differential equation is

$$(1-\mu^2)\frac{d^2T}{d\mu^2}-2\mu\frac{dT}{d\mu}+\alpha(\alpha+1)T-\frac{s^2}{1-\mu^2}T=0, \quad . \quad . \quad . \quad (70)$$

and for any value of α, real or imaginary, this is satisfied by the integral

$$\int_0^\pi\{\mu-\cos\phi\sqrt{(\mu^2-1)}\}^\alpha\cos s\phi\,d\phi.*$$

Also, if we put

$$T_\alpha^{(s)}(\mu)=(1-\mu^2)^{s/2}\frac{d^s}{d\mu^s}\{P_\alpha(\mu)\},$$

* HEINE, 'Handbuch der Kugelfunctionen,' pp. 225 *et seq.*

$P_\alpha(\mu)$ is a solution of the differential equation of zonal harmonics

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP}{d\mu}\right\}+\alpha(\alpha+1)P=0, \qquad (71)$$

and this is satisfied by the integral

$$\int_0^\pi \{\mu-\cos\phi\sqrt{(\mu^2-1)}\}^\alpha d\phi.$$

This form would not be adapted for arithmetical work if α were imaginary.

If α be imaginary, then will $\alpha(\alpha+1)=-(\frac{1}{4}+q^2)$, where q is real. If q be integral, (71) is the equation of MEHLER's 'Kegelfunctionen'; and it is shown by NEUMANN* that this equation is satisfied by the integral

$$\int_0^\infty \frac{\cos q\phi \, d\phi}{\sqrt{(\mu+\cosh\phi)}},$$

and this is finite when $\mu=1$, but infinite when $\mu=-1$; the form of demonstration adopted holds equally when q is not integral.

In general, writing $-\omega=\alpha(\alpha+1)$, and changing the independent variable to $z=(1-\mu)/2$, the equation for P becomes

$$\frac{d^2P}{dz^2}+\frac{1-2z}{z(1-z)}\frac{dP}{dz}-\frac{\omega}{z(1-z)}P=0,$$

so that one solution for P is the hypergeometric series $F(\alpha', \beta', \gamma', z)$ where $\alpha'+\beta'=1$, $\alpha'\beta'=\omega$, $\gamma'=1$; and this is finite for $z=0$ or $\mu=1$, so that

$$P_\alpha(\mu)=1+\omega\frac{1-\mu}{2}+\frac{\omega(\omega+2)}{2!\,2!}\left(\frac{1-\mu}{2}\right)^2+\ldots\frac{\omega(\omega+1.2)(\omega+2.3)\ldots(\omega+\overline{r-1}.r)}{r!\;r!}\left(\frac{1-\mu}{2}\right)^r+\ldots,$$

which converges for all real values of μ between $+1$ and -1, but diverges for $\mu=-1$.

In our equations the quantity β is always real; the quantity α may be complex of the form $-\frac{1}{2}+\iota q$; in any case we have always a solution of our equations in series or definite integrals.

18. Supposing $T_\alpha^{(s)}(\mu)$, $T_\beta^{(s)}(\mu)$ known, we shall be able to write down the values of σ_1, σ_2, ϖ; and then, supposing the surface bounded by a small circle $\mu=$ const., we have for the boundary-conditions

* "Ueber die Mehler'schen Kegelfunctionen," 'Mathemat. Annalen,' vol. 18, 1881.

$$\left.\begin{aligned} \sigma_1 + \sigma\,\sigma_2 &= 0, \\ \varpi &= 0, \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (72)$$

for some fixed value of μ.

Returning to our equations (69), we have, omitting the ϕ factors,

$$a\sigma_1 = w + \frac{\partial u}{\partial \theta} = w - \sqrt{(1-\mu^2)}\frac{du}{d\mu},$$

$$a\sigma_2 = w + u\cot\theta + \frac{sv}{\sin\theta} = w + \frac{u\mu + sv}{\sqrt{(1-\mu^2)}},$$

$$a\,\varpi = -\frac{su}{\sin\theta} + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{v}{\sin\theta}\right) = -\frac{su}{\sqrt{(1-\mu^2)}} - (1-\mu^2)\frac{d}{d\mu}\left(\frac{v}{\sqrt{(1-\mu^2)}}\right);$$

hence,

$$a\sigma_1 = \left\{\mathrm{A}\mathrm{T}_\alpha^{(s)}(\mu) - \sqrt{(1-\mu^2)}\frac{d}{d\mu}\left[\lambda\mathrm{A}\sqrt{(1-\mu^2)}\frac{d\mathrm{T}_\alpha^{(s)}(\mu)}{d\mu} + \frac{\mathrm{B}}{\sqrt{(1-\mu^2)}}\mathrm{T}_\beta^{(s)}(\mu)\right]\right\}\cos s\phi\, e^{\iota pt},$$

$$a\sigma_2 = \left\{\mathrm{A}\mathrm{T}_\alpha^{(s)}(\mu) + \frac{\mu}{\sqrt{(1-\mu^2)}}\left[\lambda\mathrm{A}\sqrt{(1-\mu^2)}\frac{d\mathrm{T}_\alpha^{(s)}(\mu)}{d\mu} + \frac{\mathrm{B}}{\sqrt{(1-\mu^2)}}\mathrm{T}_\beta^{(s)}(\mu)\right]\right.$$
$$\left. + \frac{s}{\sqrt{(1-\mu^2)}}\left[\frac{s\lambda\mathrm{A}}{\sqrt{(1-\mu^2)}}\mathrm{T}_\alpha^{(s)}(\mu) + \frac{\mathrm{B}}{s}\sqrt{(1-\mu^2)}\frac{d\mathrm{T}_\beta^{(s)}(\mu)}{d\mu}\right]\right\}\cos s\phi\, e^{\iota pt},$$

$$-a\varpi = \left\{\frac{s}{\sqrt{(1-\mu^2)}}\left[\lambda\mathrm{A}\sqrt{(1-\mu^2)}\frac{d\mathrm{T}_\alpha^{(s)}(\mu)}{d\mu} + \frac{\mathrm{B}}{\sqrt{(1-\mu^2)}}\mathrm{T}_\beta^{(s)}(\mu)\right]\right.$$
$$\left. + (1-\mu^2)\frac{d}{d\mu}\left[\frac{s\lambda\mathrm{A}}{\sqrt{(1-\mu^2)}}\mathrm{T}_\alpha^{(s)}(\mu) + \frac{\mathrm{B}}{s}\frac{d\mathrm{T}_\beta^{(s)}(\mu)}{d\mu}\right]\right\}\sin s\phi\, e^{\iota pt};$$

or

$$\left.\begin{aligned} a\sigma_1 &= \mathrm{A}\left[\left(1 + \frac{\kappa^2}{2c} - \frac{s^2\lambda}{1-\mu^2}\right)\mathrm{T}_\alpha - \lambda\mu\frac{d\mathrm{T}_\alpha}{d\mu}\right] - \mathrm{B}\left[\frac{\mu}{1-\mu^2}\mathrm{T}_\beta + \frac{d\mathrm{T}_\beta}{d\mu}\right], \\ a\sigma_2 &= \mathrm{A}\left[\left(1 + \frac{s^2\lambda}{1-\mu^2}\right)\mathrm{T}_\alpha + \lambda\mu\frac{d\mathrm{T}_\alpha}{d\mu}\right] + \mathrm{B}\left[\frac{\mu}{1-\mu^2}\mathrm{T}_\beta + \frac{d\mathrm{T}_\beta}{d\mu}\right], \\ a\,\varpi &= -2s\lambda\mathrm{A}\left[\frac{\mu}{1-\mu^2}\mathrm{T}_\alpha + \frac{d\mathrm{T}_\alpha}{d\mu}\right] + \frac{\mathrm{B}}{s}\left[\beta(\beta+1)\mathrm{T}_\beta - 2\mu\frac{d\mathrm{T}_\beta}{d\mu}\right], \end{aligned}\right\} \quad (73)$$

omitting ϕ and t factors, and writing T_α and T_β for $\mathrm{T}_\alpha^{(s)}(\mu)$, $\mathrm{T}_\beta^{(s)}(\mu)$.

Substituting in the boundary-equations (72), we have, on elimination of the ratio A : B, the frequency-equation

$$\left\{\left[1 + \sigma + \frac{\kappa^2}{2c} - \frac{s^2\lambda}{1-\mu^2}(1-\sigma)\right]\mathrm{T}_\alpha - \lambda(1-\sigma)\mu\frac{d\mathrm{T}_\alpha}{d\mu}\right\}\left\{\beta(\beta+1)\mathrm{T}_\beta - 2\mu\frac{d\mathrm{T}_\beta}{d\mu}\right\}$$
$$= 2s^2\lambda(1-\sigma)\left(\frac{\mu}{1-\mu^2}\mathrm{T}_\alpha + \frac{d\mathrm{T}_\alpha}{d\mu}\right)\left(\frac{\mu}{1-\mu^2}\mathrm{T}_\beta + \frac{d\mathrm{T}_\beta}{d\mu}\right); \quad \ldots \quad (74)$$

and, if $\mu = 0$ at the edge, or the shell be hemispherical, this is

$$\mathrm{T}_\beta . \mathrm{T}_\alpha \left[1 + \sigma + \frac{\kappa^2}{2c} - s^2 \lambda (1 - \sigma) \right] \beta (\beta + 1) = 2 s^2 \lambda (1 - \sigma) \frac{d\mathrm{T}_\alpha}{d\mu} \cdot \frac{d\mathrm{T}_\beta}{d\mu} . \quad . \quad (75)$$

In the case of the symmetrical vibrations $s = 0$ and the expressions found involve indeterminates. In any other case the above expressions show that it will not be possible for the motion to be purely tangential, since, for this, $\mathrm{A} = 0$, and we should have to make $d\mathrm{T}_\beta / d\mu = 0$, $\mathrm{T}_\beta = 0$ for some value of μ.*

19. In the case of the symmetrical vibrations we have to put u, v, w independent of ϕ in (54), (55), (56); this gives

$$\left.\begin{aligned} &\frac{d^2 u}{d\theta^2} + \cot\theta \frac{du}{d\theta} + \left(\frac{2 + \kappa^2}{1 + c} - \frac{1}{\sin^2\theta} \right) u = 0, \\ &\frac{d^2 v}{d\theta^2} + \cot\theta \frac{dv}{d\theta} + \left(2 + \kappa^2 - \frac{1}{\sin^2\theta} \right) v = 0, \\ &\frac{\kappa^2}{2c} w = \frac{du}{d\theta} + u \cot\theta . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

From which

$$\left.\begin{aligned} u &= \mathrm{A} \sqrt{(1 - \mu^2)} \frac{d\mathrm{P}_\alpha(\mu)}{d\mu}, \\ v &= \mathrm{B} \sqrt{(1 - \mu^2)} \frac{d\mathrm{P}_\beta(\mu)}{d\mu}, \\ w &= \frac{2c}{\kappa^2} \alpha (\alpha + 1) \mathrm{A} \mathrm{P}_\alpha(\mu), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

where β, α have the same meaning as before.

Hence,

$$\left.\begin{aligned} a\sigma_1 &= \mathrm{A} \left[\left(1 + \frac{2c}{\kappa^2} \right) \alpha (\alpha + 1) \mathrm{P}_\alpha - \mu \frac{d\mathrm{P}_\alpha}{d\mu} \right], \\ a\sigma_2 &= \mathrm{A} \left[\frac{2c}{\kappa^2} \alpha (\alpha + 1) \mathrm{P}_\alpha + \mu \frac{d\mathrm{P}_\alpha}{d\mu} \right], \\ a\varpi &= \mathrm{B} \left[\beta (\beta + 1) \mathrm{P}_\beta - 2\mu \frac{d\mathrm{P}_\beta}{d\mu} \right]. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

The boundary-equations (72) become

$$\left.\begin{aligned} \mathrm{A} \left[\left\{ 1 + (1 + \sigma) \frac{2c}{\kappa^2} \right\} \alpha (\alpha + 1) \mathrm{P}_\alpha - (1 - \sigma) \mu \frac{d\mathrm{P}_\alpha}{d\mu} \right] &= 0, \\ \mathrm{B} \left[\beta (\beta + 1) \mathrm{P}_\beta - 2\mu \frac{d\mathrm{P}_\beta}{d\mu} \right] &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (79)$$

* Using only the differential equations, MATHIEU supposed that there could be unsymmetrical tangential vibrations.

which can be satisfied either by

$$B = 0, \qquad \text{and} \qquad \left\{1 + \frac{2c}{\kappa^2}(1+\sigma)\right\}\alpha(\alpha+1)\,P_\alpha - (1-\sigma)\,\mu\,\frac{dP_\alpha}{d\mu} = 0, \qquad (80)$$

or by

$$A = 0, \qquad \text{and} \qquad \beta(\beta+1)\,P_\beta - 2\mu\,\frac{dP_\beta}{d\mu} = 0 \qquad (81)$$

This gives two types of motion.

In the first the motion is partly tangential and partly radial. Since $P_\alpha(\mu)$ cannot have equal roots, u and w cannot vanish together, or there are no lines of no displacement. The displacement is purely radial along the lines $dP_\alpha(\mu)/d\mu = 0$, and purely tangential along the lines $P_\alpha(\mu) = 0$. The ratios of the frequencies of the component vibrations of this type depend on σ, *i.e.*, on the material of the shell.

In the second type the motion is purely tangential, every point moving through a distance along the parallel to the edge through it, which is the same at all points of the parallel. The lines $dP_\beta(\mu)/d\mu = 0$ are nodal. The ratios of the frequencies of the component vibrations are independent of the material of the shell.

20. For a hemispherical bowl $\mu = 0$ at the edge.

(1.) In the motions of the first type $P_\alpha(\mu)$ is to vanish with μ; hence, α is an odd integer, or, i being any integer, we have

$$(2+\kappa^2)/(1+c) = (2i+1)(2i+2) = \omega \text{ say,}$$

where

$$c\left[\kappa^2 - 4(1+\sigma)/(1-\sigma)\right] = \kappa^2(1+\sigma)/(1-\sigma).$$

This gives

$$\kappa^4(1-\sigma) - 2\kappa^2(1+3\sigma+\omega) + 4(\omega-2)(1+\sigma) = 0; \qquad (82)$$

this equation has always real roots.

If κ_i^2, $\kappa_i'^2$ be the roots, and p_i, p'_i the corresponding values of p, according to the formula $p^2a^2\rho = n\kappa^2$, then

$$\left.\begin{aligned} u &= \sum_{i=0}^{i=\infty}\left[\sqrt{(1-\mu^2)}\,\frac{d}{d\mu}\{P_{2i+1}(\mu)\}\{A_i e^{\iota p_i t} + A'_i e^{\iota p'_i t}\}\right], \\ v &= 0, \\ w &= \sum_{i=0}^{i=\infty}\left[4\sqrt{(1-\mu^2)}(i+1)(2i+1)\,P_{2i+1}(\mu)\left\{\frac{c_i}{\kappa_i^2}A_i e^{\iota p_i t} + \frac{c'_i}{\kappa'^2_i}A'_i e^{\iota p'_i t}\right\}\right]. \end{aligned}\right\} \qquad (83)$$

To get arithmetical results, let us choose $\sigma = \frac{1}{3}$; the equation for κ^2 becomes

$$\kappa^4 - 6\{1 + (i+1)(2i+1)\}\kappa^2 + 8\{(2i+1)(2i+2) - 2\} = 0,$$

and κ_i, κ'_i are given by the table:—

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i	0	1	2	3	4
$6\{(1+(i+1)(2i+1)\}$	12	42	96	174	276
$8\{(2i+1)(2i+2)-2\}$	0	80	224	432	704
κ^2	12	$\begin{cases}40\\2\end{cases}$	$48 \pm \surd(2080)$	$87 \pm \surd(7137)$	$138 \pm \surd(18340)$
κ_i	3·464 . . .	6·324 . . .	9·676 . . .	13·095 . . .	16·505 . . .
κ'_i	0	1·414 . . .	1·535 . . .	1·587 . . .	1·604 . . .

The tones of the second series are all near together; those of the first are separated by intervals rather less than for a harmonic scale.

(2.) In the motions of the second type A = 0, and $P_\beta(\mu)$ vanishes with μ; hence β is an odd number, and

$$2+\kappa^2=(2i+1)(2i+2)=\omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

If p''_i be the value of p corresponding to κ''_i, we have

and

$$\left.\begin{array}{l} u=0, \qquad w=0, \\ v=\sum_{i=1}^{i=\infty}\left[\surd(1-\mu^2)\dfrac{d}{d\mu}\{P_{2i+1}(\mu)\}B_i e^{\iota p''_i t}\right]; \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (85)$$

and κ''_i is given by the table:—

i	1	2	3	4	5
ω_i-2	10	28	54	88	130
κ''_i	3·158	5·291	7·347	9·380	11·401
$\kappa''_i : \kappa''_1$	1	1·673	2·323	2·966	3·605 . . .

These intervals are nearly fifths.

§ 8. *Vibrations of Cylindrical Shell.*

21. As a further example, suppose the middle-surface of the shell cylindrical; and, to fix ideas, suppose there is a rigid disc at one end, and at a distance c from it a free edge bounded by a circle.

Let a be the radius of the circular section of the cylinder, and a, z, ϕ cylindrical coordinates of a point on the middle-surface, the origin being at the centre of the rigid disc.

In the equations of motion of Art. 13, we have to put

$$h_1 = 1, \quad h_2 = 1/a, \quad 1/\rho_1 = 0, \quad 1/\rho_2 = 1/a.$$

Taking u, v, w proportional to e^{ipt}, and $\kappa^2 n = p^2 a^2 \rho$, these equations become

$$\frac{\partial^2 u}{\partial z^2} + \frac{\kappa^2}{a^2} u + \frac{1}{a^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{3m-n}{m+n}\left(\frac{\partial^2 u}{\partial z^2} + \frac{1}{a}\frac{\partial^2 v}{\partial z\,\partial \phi}\right) + 2\frac{m-n}{m+n}\frac{1}{a}\frac{\partial w}{\partial z} = 0, \quad . \quad . \quad . \quad (86)$$

$$\frac{\partial^2 v}{\partial z^2} + \frac{\kappa^2}{a^2} v + \frac{1}{a^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{3m-n}{m+n}\left(\frac{1}{a^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{1}{a}\frac{\partial^2 u}{\partial z\,\partial \phi}\right) + 4\frac{m}{m+n}\frac{1}{a^2}\frac{\partial w}{\partial \phi} = 0, \quad . \quad . \quad (87)$$

$$\frac{\kappa^2}{a^2} w = \frac{4m}{m+n}\frac{1}{a^2}\left(w + \frac{\partial v}{\partial \phi}\right) + 2\frac{m-n}{m+n}\frac{1}{a}\frac{\partial u}{\partial z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

Put $4m\beta/(m+n) = \kappa^2 - 4m/(m+n)$, then (88) is

$$\beta w = \frac{\partial v}{\partial \phi} + \sigma a \frac{\partial u}{\partial z}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

and (86), (87) give

$$\frac{4m}{m+n}\frac{\partial^2 u}{\partial z^2} + \frac{\kappa^2}{a^2} u + \frac{1}{a^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{3m-n}{m+n}\frac{1}{a}\frac{\partial^2 v}{\partial z\,\partial \phi} + \frac{2}{a\beta}\frac{m-n}{m+n}\left(\frac{\partial^2 v}{\partial z\,\partial \phi} + \sigma a\frac{\partial^2 u}{\partial z^2}\right) = 0,$$

$$\frac{\partial^2 v}{\partial z^2} + \frac{\kappa^2}{a^2} v + \frac{4m}{m+n}\frac{1}{a^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{3m-n}{m+n}\frac{1}{a}\frac{\partial^2 u}{\partial z\,\partial \phi} + \frac{1}{a^2\beta}\frac{4m}{m+n}\left(\sigma a\frac{\partial^2 u}{\partial z\,\partial \phi} + \frac{\partial^2 v}{\partial \phi^2}\right) = 0,$$

or

$$\frac{\partial^2 u}{\partial z^2}\left(\frac{4m}{m+n} + \frac{2\sigma}{\beta}\frac{m-n}{m+n}\right) + \frac{1}{a^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\kappa^2}{a^2} u + \frac{1}{a}\left[1 + 2\frac{m-n}{m+n}\left(1 + \frac{1}{\beta}\right)\right]\frac{\partial^2 v}{\partial z\,\partial \phi} = 0, \quad (90)$$

$$\frac{\partial^2 v}{\partial z^2} + \frac{4m}{m+n}\left(1 + \frac{1}{\beta}\right)\frac{1}{a^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{\kappa^2}{a^2} v + \frac{1}{a}\left[1 + 2\frac{m-n}{m+n}\left(1 + \frac{1}{\beta}\right)\right]\frac{\partial^2 u}{\partial z\,\partial \phi} = 0. \quad . \quad (91)$$

Let $u \propto \cos s\phi$, $v \propto \sin s\phi$; then for u and v as functions of z we have the equations

$$\left.\begin{aligned} A\frac{d^2u}{dz^2} + Bu + C\frac{dv}{dz} &= 0, \\ \frac{d^2v}{dz^2} + Dv - C\frac{du}{dz} &= 0, \end{aligned}\right\} \quad \quad (92)$$

where

$$\left.\begin{aligned} A &= \frac{2}{1-\sigma}\left(1+\frac{\sigma^2}{\beta}\right), \\ B &= \frac{\kappa^2 - s^2}{a^2}, \\ C &= \frac{s}{a}\left(1 + \frac{2\sigma}{1-\sigma}\cdot\frac{1+\beta}{\beta}\right), \\ D &= \frac{1}{a^2}\left(\kappa^2 - s^2\frac{1+\beta}{\beta}\,\frac{2}{1-\sigma}\right). \end{aligned}\right\} \quad \quad (93)$$

To satisfy these, take

$$\left.\begin{aligned} u &= P\cos\mu z \\ v &= Q\sin\mu z \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} u &= -P'\sin\mu z \\ v &= \quad Q'\cos\mu z \end{aligned}\right\} \quad \quad (94)$$

Then

$$\left.\begin{aligned} P(B - A\mu^2) + QC\mu &= 0, \\ PC\mu + Q(D - \mu^2) &= 0; \end{aligned}\right\}$$

$$\left.\begin{aligned} P'(B - A\mu^2) + Q'C\mu &= 0, \\ P'C\mu + Q'(D - \mu^2) &= 0, \end{aligned}\right\}$$

whence

$$(B - A\mu^2)(D - \mu^2) - C^2\mu^2 = 0.$$

This is a quadratic in μ^2, viz.:

$$A\mu^4 - (B + AD + C^2)\mu^2 + BD = 0; \quad \quad (95)$$

and we have

$$Q = C.P.\,\frac{\mu}{\mu^2 - D}, \qquad Q' = C.P'.\,\frac{\mu}{\mu^2 - D}. \quad \quad (96)$$

Let μ_1^2, μ_2^2 be the roots of the quadratic (95), then

$$\left.\begin{aligned} u &= \cos s\phi\, e^{\iota pt}[P_1\cos\mu_1 z + P_2\cos\mu_2 z - P'_1\sin\mu_1 z - P'_2\sin\mu_2 z], \\ v &= \sin s\phi\, e^{\iota pt}[Q_1\sin\mu_1 z + Q_2\sin\mu_2 z + Q'_1\cos\mu_1 z + Q'_2\cos\mu_2 z], \end{aligned}\right\} \quad (97)$$

so that

$$\left.\begin{aligned} \beta w = {} & \cos s\phi\, e^{\iota pt}[(sQ_1 - \sigma a\mu_1 P_1)\sin\mu_1 z + (sQ_2 - \sigma a\mu_2 P_2)\sin\mu_2 z] \\ & + \cos s\phi\, e^{\iota pt}[(sQ'_1 - \sigma a\mu_1 P'_1)\cos\mu_1 z + (sQ'_2 - \sigma a\mu_2 P'_2\cos)\,\mu_2 z]\cdot \end{aligned}\right\} \quad (98)$$

Calculating from these, we find, dropping the time-factor,

$$\left.\begin{aligned}
\sigma_1 &= -\cos s\phi\,[\mu_1 P_1 \sin\mu_1 z + \mu_2 P_2 \sin\mu_2 z + \mu_1 P'_1 \cos\mu_1 z + \mu_2 P'_2 \cos\mu_2 z],\\
\sigma_2 &= \cos s\phi\left[\left(\frac{s}{a}\frac{\beta+1}{\beta}Q_1 - \frac{\sigma\mu_1}{\beta}P_1\right)\sin\mu_1 z + \left(\frac{s}{a}\frac{\beta+1}{\beta}Q_2 - \frac{\sigma\mu_2}{\beta}P_2\right)\sin\mu_2 z\right]\\
&\quad + \cos s\phi\left[\left(\frac{s}{a}\frac{\beta+1}{\beta}Q'_1 - \frac{\sigma\mu_1}{\beta}P'_1\right)\cos\mu_1 z + \left(\frac{s}{a}\frac{\beta+1}{\beta}Q'_2 - \frac{\sigma\mu_2}{\beta}P'_2\right)\cos\mu_2 z\right],\\
\varpi &= \sin s\phi\left[\left(\mu_1 Q_1 - \frac{s}{a}P_1\right)\cos\mu_1 z + \left(\mu_2 Q_2 - \frac{s}{a}P_2\right)\cos\mu_2 z\right]\\
&\quad - \sin s\phi\left[\left(\mu_1 Q'_1 - \frac{s}{a}P'_1\right)\sin\mu_1 z + \left(\mu_2 Q'_2 - \frac{s}{a}P'_2\right)\sin\mu_2 z\right].
\end{aligned}\right\} \quad (99)$$

If there is a rigid disc at $z = 0$, then v and w vanish with z, so that

$$\left.\begin{aligned} Q'_1 + Q'_2 &= 0,\\ \mu_1 P'_1 + \mu_2 P'_2 &= 0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

The first of these is, by (96),

$$\frac{\mu_1 P'_1}{\mu_1^2 - D} + \frac{\mu_2 P'_2}{\mu_2^2 - D} = 0,$$

so that (100) can only be satisfied by P'_1, P'_2, both zero, and consequently Q'_1, Q'_2, both zero, unless we take $\mu_1^2 = \mu_2^2$ and $Q'_1 + Q'_2 = 0$.

If $\mu_1^2 = \mu_2^2$, we have $P'_1 = \mp P'_2$ and $Q'_1 = -Q'_2$, so that the terms in u, v, w, σ_1, σ_2, ϖ which contain P'_1, P'_2, Q'_1, Q'_2 all vanish identically.

It follows that to satisfy the conditions at $z = 0$ we must drop out the P', Q' terms.

The boundary-conditions at $z = c$ are—

$$\left.\begin{aligned} \sigma_1 + \sigma\sigma_2 &= 0,\\ \varpi &= 0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

where we have to take only the part in P_1, P_2, Q_1, Q_2 and to write

$$Q_1 = \frac{C\mu_1}{\mu_1^2 - D}P_1, \qquad Q_2 = \frac{C\mu_2}{\mu_2^2 - D}P_2.$$

Hence, we have

$$\mu_1 P_1 \sin\mu_1 c\left[1 + \frac{\sigma^2}{\beta} - \frac{\sigma s}{a}\frac{\beta+1}{\beta}\frac{C}{\mu_1^2 - D}\right] + \mu_2 P_2 \sin\mu_2 c\left[1 + \frac{\sigma^2}{\beta} - \frac{\sigma s}{a}\frac{\beta+1}{\beta}\frac{C}{\mu_2^2 - D}\right] = 0,$$

and

$$P_1 \cos\mu_1 c\left(\frac{s}{a} - \frac{\mu_1^2 C}{\mu_1^2 - D}\right) + P_2 \cos\mu_2 c\left(\frac{s}{a} - \frac{\mu_2^2 C}{\mu_2^2 - D}\right) = 0.$$

Eliminating P_1, P_2, we get

$$\mu_1 \sin \mu_1 c \cos \mu_2 c \left(\frac{s}{a} - \frac{\mu_2^2 C}{\mu_2^2 - D}\right)\left(1 + \frac{\sigma^2}{\beta} - \frac{\sigma s}{a}\frac{\beta+1}{\beta}\frac{C}{\mu_1^2 - D}\right)$$

$$= \mu_2 \sin \mu_2 c \cos \mu_1 c \left(\frac{s}{a} - \frac{\mu_1^2 C}{\mu_1^2 - D}\right)\left(1 + \frac{\sigma^2}{\beta} - \frac{\sigma s}{a}\frac{\beta+1}{\beta}\frac{C}{\mu_2^2 - D}\right),$$

or

$$\sin(\mu_1 + \mu_2) c \left[C^2 \frac{\mu_1\mu_2(\mu_2 - \mu_1)}{(\mu_1^2 - D)(\mu_2^2 - D)}\frac{\sigma s}{a}\frac{\beta+1}{\beta} - C\frac{\sigma s^2}{a}\frac{\beta+1}{\beta}\left(\frac{\mu_1}{\mu_1^2 - D} - \frac{\mu_2}{\mu_2^2 - D}\right)\right.$$
$$\left. + C\mu_1\mu_2\left(1 + \frac{\sigma^2}{\beta}\right)\left(\frac{\mu_1}{\mu_1^2 - D} - \frac{\mu_2}{\mu_2^2 - D}\right) + \frac{s}{a}\left(1 + \frac{\sigma^2}{\beta}\right)(\mu_1 - \mu_2)\right]$$
$$+ \sin(\mu_1 - \mu_2) c \left[C^2 \frac{\mu_1\mu_2(\mu_1 + \mu_2)}{(\mu_1^2 - D)(\mu_2^2 - D)}\frac{\sigma s}{a}\frac{\beta+1}{\beta} - C\frac{\sigma s^2}{a}\frac{\beta+1}{\beta}\left(\frac{\mu_1}{\mu_1^2 - D} + \frac{\mu_2}{\mu_2^2 - D}\right)\right.$$
$$\left. - C\mu_1\mu_2\left(1 + \frac{\sigma^2}{\beta}\right)\left(\frac{\mu_1}{\mu_1^2 + D} + \frac{\mu_2}{\mu_2^2 + D}\right) + \frac{s}{a}\left(1 + \frac{\sigma^2}{\beta}\right)(\mu_1 + \mu_2)\right] = 0.$$

From (95), $A(\mu_1^2 - D)(\mu_2^2 - D) = AD^2 - D(B + AD + C^2) + BD = -DC^2$; substituting and re-arranging, we find

$$\frac{\sin(\mu_1 + \mu_2) c}{\mu_1 + \mu_2}\left[AC\mu_1\mu_2\frac{\sigma s}{a}\frac{\beta+1}{\beta} - CD\frac{s}{a}\frac{\beta+s^2}{\beta} - A(\mu_1\mu_2 + D)\left(\frac{\sigma s^2}{a}\frac{\beta+1}{\beta} - \mu_1\mu_2\frac{\beta+s^2}{\beta}\right)\right]$$
$$= \frac{\sin(\mu_1 - \mu_2) c}{\mu_1 - \mu_2}\left[AC\mu_1\mu_2\frac{\sigma s}{a}\frac{\beta+1}{\beta} + CD\frac{s}{a}\frac{\beta+s^2}{\beta} - A(\mu_1\mu_2 - D)\left(\frac{\sigma s^2}{a}\frac{\beta+1}{\beta} - \mu_1\mu_2\frac{\beta+s^2}{\beta}\right)\right]; \quad (102)$$

this equation gives the frequency.

22. In the case of the symmetrical vibrations, $s = 0$, and we have

$$\mu_1 = \sqrt{B/A}, \qquad \mu_2 = \sqrt{D},$$

and $P_2 = 0$, $Q_1 = 0$, but Q_2 is finite. Thus, the equation just written involves some indeterminates.

We take the solutions

$$u = P_1 \cos \mu_1 z,$$
$$v = Q_2 \sin \mu_2 z.$$

The conditions $\varpi = 0$, $\sigma_1 + \sigma\sigma_2 = 0$ reduce to

$$\left(1 + \frac{\sigma^2}{\beta}\right)\mu_1 P_1 \sin \mu_1 c = 0, \qquad Q_2\mu_2 \cos \mu_2 c = 0;$$

hence, either

$$Q_2 = 0, \quad \text{and} \quad \sin \mu_1 c = 0,$$

or

$$P_1 = 0, \quad \text{and} \quad \cos \mu_2 c = 0.$$

This gives two types of motion.

In the first, the motion is partly tangential and partly radial; the expressions for the displacements are

$$\left.\begin{aligned} u &= \sum_{i=1}^{i=\infty} \mathrm{P}_i \cos \frac{i\pi z}{c}\, e^{\iota p_i t}, \\ v &= 0, \\ w &= \sum_i \frac{-\sigma}{\kappa^2 \frac{1-\sigma}{2} - 1} \frac{a\pi i}{c} \mathrm{P}_i \sin \frac{i\pi z}{c}\, e^{\iota p_i t}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

where the equation for κ is $\frac{\mathrm{B}}{\mathrm{A}} c^2 = i^2\pi^2$, or

$$\frac{c^2}{a^2} \kappa^2 \frac{\kappa^2 (1-\sigma) - 2}{\kappa^2 - (1+\sigma) 2} = 2i^2\pi^2 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

and $p^2a^2\rho = n\kappa^2$, i being any integer.

The displacement is, for each normal type of vibration, wholly tangential along the circles $\sin i\pi z/c = 0$, and wholly radial along the circles $\cos i\pi z/c = 0$; there are no points or lines of no displacement. The frequency depends on the length and radius of the shell, and the ratios of the intervals for consecutive tones depends on σ, *i.e.*, on the material of the shell.

In the motions of the second type the displacement is purely tangential, and is expressed by

$$\left.\begin{aligned} u &= 0, \\ v &= \sum_{i=0}^{i=\infty} \mathrm{Q}_i \sin \frac{2i+1}{2} \frac{\pi z}{c}\, e^{\iota p_i t}, \\ w &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

where the equation for the frequency is

$$4\kappa^2 c^2 = (2i+1)^2 \pi^2 a^2,$$

or

$$4p_i^2 = (2i+1)^2 \pi^2 n/c^2\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

In this case the circles $\sin (2i+1) \pi z/2c = 0$, are nodal lines. The frequency varies inversely as the length of the cylinder, and the intervals between consecutive tones are independent of the material of the shell.

Note.—July, 1888.—In the paper as read some examples were next given of the application of the method to problems of equilibrium. These are now withdrawn, as of little physical interest, and not directly relevant to the subject of the paper (see Summary).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

§ 9. *Summary.**

This paper is really an attempt to construct a theory of the vibrations of bells. In any actual bell complications will arise, which have been omitted in this discussion, partly from variations of the thickness in different parts, and partly from the want of isotropy in the material. We can hardly expect a metal which has been subjected to the process of bell-manufacture to be other than very æolotropic, while it is notorious that bells are usually thickest at the rim. The difficulty of the problem in its general form seems to make it advisable to begin with the limiting case of an indefinitely thin perfectly isotropic shell, whose thickness is everywhere constant, and so small compared with its linear dimensions, that powers of it above the first may be neglected in mathematical expressions, which contain the first and higher powers multiplied by quantities of the same order of magnitude.

Of previous theoretical work we have examples in Lord RAYLEIGH'S 'Theory of Sound,' and in his paper on the "Bending of Surfaces of Revolution," in ARON'S and MATHIEU'S memoirs, and in IBBETSON'S treatise on the Mathematical Theory of Elasticity. In the 'Theory of Sound' Lord RAYLEIGH treats the vibrations of a thin ring or infinite cylinder of matter, supposed to be deformed in such a way that the motion is in one plane and the elements remain unextended, and remarks that at the time of publication this was the nearest approximation to a theoretical treatment of bells. He afterwards applies his theory of the bending of surfaces to obtain a more exact analytical method of treating the problem, but his disregard of the boundary-conditions which hold at a free edge appears to vitiate this theory. ARON can hardly be said to have attained a theory of bells, and the interest of his memoir is mainly mathematical; his inaccuracies have been already referred to. I have also previously referred to the objection which lies against MATHIEU'S method of treatment; this and the complexity and difficulty of some of his analysis seem to render a new method desirable. I shall have to refer to IBBETSON later.

The theory here put forward rests on the form of the function expressing the potential-energy of deformation per unit area of the middle-surface of the shell. Supposing that the surface is stretched and has its curvature changed, we find that the energy consists of two terms. One of these contains only the functions defining the stretching, while the other contains also those defining the bending of the middle-surface. The modulus of stretching is proportional to the thickness, while the modulus of bending is proportional to its cube. Unless, therefore, the functions expressing the stretching, viz., the extensions and shear of rectangular line-elements of the middle-surface, are of a higher order of small quantities than those defining the bending, viz., the changes of the principal curvatures and of the directions of the principal planes, the vibrations depend on the term which involves the stretching, and not on that which involves the bending. Now, it seems to have been universally

* Partly rewritten, July, 1888.

assumed by English writers that the reverse of this is the case, viz., that the vibrations take place in such a way that no line on the middle-surface is altered in length. This will be borne out by a reference to Lord RAYLEIGH and IBBETSON. The theory of the present paper rests on the fact that the functions expressing the stretching and those expressing the changes in magnitude and direction of curvature are of the same order of small quantities. This is proved in the following way:—The potential energy consists of two parts; one, Q_2, proportional to the thickness h; and the other, Q_1, proportional to h^3. The first is expressed in terms of the stretching, and the second in terms of the bending of the middle-surface. Some previous theories have proceeded as if Q_1 only occurred. If this were the case, we ought to get an approximation by supposing that $Q_2/h = 0$. This is equivalent to assuming that there is no stretching of the middle-surface. We should therefore get an approximation by supposing the surface inextensible to the first order. The stretching and the bending are expressed, to the first order, by linear functions of certain differential coefficients of the displacements. Our supposed method of getting an approximation is then to make the functions expressing the stretching vanish. Now, I have shown that the functions expressing the displacement are thus, to a certain extent, determined, and *that* in such a way that the boundary-conditions cannot be satisfied. The boundary-conditions referred to are the exact conditions found by retaining the complete expression for the potential energy. It is inferred that the functions expressing the stretching cannot be taken equal to zero for an approximation; or, in other words, small compared with those expressing the bending; and, thus, Q_1/h^3 and Q_2/h, are of the same order of magnitude. The conclusion that Q_1 is small compared with Q_2 seems inevitable.

The argument breaks down for a plane plate through the vanishing of the curvatures; Q_1 is then alone of importance. In the case of an open shell or bowl whose linear dimension is small compared with its radius of curvature, and large compared with its thickness, both terms are important. When this is so, we get a class of cases for which the linear dimensions concerned are of three different orders of magnitude, and this case will not come under the method of the present paper. It may be compared with the problem of the watch-spring mentioned in THOMSON and TAIT'S 'Natural Philosophy,' Part 2, p. 264, which stands between a bar and a plate. The very open shell or bowl stands in the same way between a plate and what I have called a shell.

The theory of this paper proceeds as if Q_2 alone occurred. It is to be regarded as the limiting form for indefinitely thin shells. A complete theory of bells, even when regarded as uniformly thick and isotropic, could only be obtained by using the exact equations formed by retaining both terms of the potential energy.

Again, English writers have assumed that the potential energy, which they suppose to depend only on the bending, will be the same quadratic function of the changes of principal curvature as it is for a plane plate. The same authorities as before may be quoted, and we may also refer to a question set in the Mathematical Tripos, January 18th, morn., 1878, question η. To test this assumption involved the investigation of Artt. 7, 8, and the result is that it is only in the case of a sphere supposed unstretched that the potential energy has this form. This is the case treated by Lord RAYLEIGH, but his method still fails, for a complete sphere cannot be bent without stretching, while, if the sphere be incomplete, the conditions which hold at a free edge cannot be satisfied; this is explicitly proved in Art. 14.

A general result is derived from the consideration of the functions expressing the kinetic and potential energies, Q_2 only being retained. Both these functions are proportional to the thickness of the shell, and thus the periods of vibration are independent of the thickness. That this result holds for a complete thin spherical shell vibrating in any manner has been demonstrated by LAMB ('London Math. Soc. Proc.,' vol. 14, 1882, p. 52). His equations (7) and (9) when reduced are independent of the thickness.

Two general results are obtained without solution from the equations of motion. The first is, that vibrations involving displacement along the normal only are impossible except in the cases of the plane, complete sphere, and infinitely long circular cylinder. IBBETSON's treatment of the problem appears to assume (1) inextensibility, (2) the incorrect formula for the energy, (3) normal displacements. The other result is that any surface of revolution can execute purely tangential vibrations which are symmetrical with respect to the axis of revolution, and in which the motion is purely torsional, or perpendicular to the planes through the axis. These must not be confounded with the familiar vibrations of finger-bowls, which are most probably a type with two nodal meridians.*

The theory of the vibrations of a thin spherical shell bounded by a small circle is an interesting example of the general theory of vibrations of an elastic solid. In an infinite solid there are two types of vibratory motion, the longitudinal and the distortional, both of which are propagated as waves. In a bounded solid this state of things is modified by reflexions at the bounding-surfaces, so that the purely longitudinal and purely tangential waves do not in general exist separately. Again, in all cases of displacement in one direction only, as in the vibrations of strings, bars, and plates, there may be displacements in different directions which are independent of each other, with their corresponding nodal lines or points. This also is modified in the general solid. The types of vibration, for example, of a portion of a spherical shell bounded by a small circle are partially made out in this essay. One immediate result is that there are in general no nodal lines, properly so called. In any type the displacement along the parallels vanishes at one set of meridians; the other displacements vanish together at another set of meridians. These sets are ranged at equal intervals round the sphere. There appears to be good reason to suppose generally that the corresponding proposition will not obtain with reference to nodal parallels. The establishment of the fact would require a solution of the general frequency equa-

* RAYLEIGH, 'Sound,' vol. 1, Art. 234.

tion, and this I have not been able to effect. One case, however, is readily solved, and that is where the displacement is symmetrical with respect to the pole of the sphere. It appears here that the vibrations divide themselves into two types, one purely tangential with displacement along the parallels, the other partly radial and partly consisting of displacements along the meridians. There are no nodal meridians. In the purely tangential vibrations there exists a series of nodal parallels, whose number corresponds to the type of vibration. The intervals for the various tones are each of them nearly a fifth. In the partly radial vibrations the radial displacement vanishes at one set of small circles, and the tangential displacement at another set. The number and position of the nodal circles for the purely tangential vibration coincide exactly with the number and position of the circles along which the tangential displacement vanishes in the corresponding partly radial mode. The vibrations of the two types belong to different normal modes of vibration, and have different frequencies. If we like to extend the meaning of "nodal lines," so as to include the small circles just referred to, then we may state another result in the form that for partly radial vibrations there are two periods and modes of vibration which have the same set of "nodal lines." The tones of one of these sets are all very near together; those of the other set are separated by intervals nearly the same as for a harmonic scale.

A discussion of the vibrations of an elastic shell in the form of a circular cylinder closed at one end by a rigid disc perpendicular to its axis leads to similar conclusions as to types of vibration and their definition by nodal lines.

It is unfortunate that solutions of the frequency equation for the case of two "nodal" meridians dividing the shell into four equal portions could not be obtained, as these probably include the gravest mode of vibration of which the shell is capable. The tones of the symmetrical vibrations discussed are very high, and the theory in its present state cannot easily be tested by experiment. There is, however, one result which would seem to admit of practical verification, viz., it is found that, for similar thin shells, the frequency is independent of the thickness, and varies inversely as the linear dimension.

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